Relative deprivation and bargaining solutions

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Abstract

Stark et al. (2012) considered a standard 2-agent pie-division problem, under the assumption that each agent cares about his own consumption as well as about if, and to what extant, he is falling behind the consumption of the other agent. They showed that the utilitarian thing to do is to split the pie equally among the agents. I show that the same conclusion is reached if instead of utilitarianism the social criterion is the Nash bargaining solution. Additionally, I discuss variants of this result and their connections to other known bargaining solutions.

Keywords: Bargaining; Nash solution; Relative deprivation.

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1 Introduction

Consider the problem of dividing a size-1 pie between two agents, agent 1 and agent 2, who have constant marginal utilities from consumption, \(\alpha_1\) and \(\alpha_2\) (both positive). The utilitarian criterion—maximization of the sum of the agents’ utilities—dictates that agent 1 be allocated the entire pie if \(\alpha_1 > \alpha_2\) while agent 2 should obtain it if the reverse strict inequality holds (when \(\alpha_1 = \alpha_2\) any non-wasteful division is utilitarian).

Now consider the following modification of the agents’ preferences: instead of deriving utility solely from his own consumption, each agent also cares about how he is doing relatively to the other agent. These preferences are captured by the utility function 
\[ u_i(x_1, x_2) = \alpha_i x_i + (1 - \alpha_i)RD_i(x_1, x_2), \]
where \(x_i\) is agent \(i\)’s pie-share, \(\alpha_i \in (0, 1)\) is a parameter, and \(RD_i\) is a relative deprivation index that measures how agent \(i\) is doing relatively to agent \(j \neq i\).\(^1\)\(^2\) Considering the relative deprivation index 
\[ RD_i(x_1, x_2) = -\frac{1}{2}\max\{x_j - x_i, 0\}, \]
Stark et al. (2012) showed that the utilitarian criterion—i.e., maximizing \(u_1 + u_2\)—dictates equal sharing, \(x_1 = x_2 = \frac{1}{2}\). Remarkably and surprisingly, this is true for any \((\alpha_1, \alpha_2) \in (0, 1)^2\).

I revisit the problem of dividing a size-1 pie between two agents whose preferences are given by the aforementioned \(u_i\)’s. However, instead of the utilitarian criterion, I take the Nash bargaining solution (Nash (1950)) to be the social objective. I show that equal sharing, \(x_1 = x_2 = \frac{1}{2}\), is the unique solution to the problem, independent of \((\alpha_1, \alpha_2)\). Therefore, egalitarian resource allocation is the right thing to do when agents assign some weight (no matter how small) to their relative standing in society; this is true not only from the standpoint of utilitarianism, but also on the basis of bargaining theory’s most prominent solution.

Section 2 presents the aforementioned result. Further discussion and results are in Sections 3-4.

\(^1\)The idea of a relative deprivation measure was proposed by Yitzhaki (1979), and is based on a classic work of Runciman (1966).

\(^2\)The standard (“selfish”) problem corresponds to \(RD_1 = RD_2 \equiv 0\).
2 The main result

According to the now-classic formulation of Nash (1950), a bargaining problem is a pair \((S, d)\), where \(S\) is the feasible set of utility allocations, from which the agents need to agree on one utility-pair, and \(d \in S\), the disagreement point, specifies their utilities in case that no agreement is reached. Under the normalization \(d = (0, 0)\), the Nash solution picks the maximizer of the (Nash) product \(s_1 \times s_2\) over \(s \in S\). Nash (1950) showed that this method for solving bargaining problems is the only one which is consistent with several appealing axioms; his solution attracted massive attention in the literature, and received further support, by many authors, and in many ways.\(^3\)

With the origin as the disagreement point, the Nash product in our context is:

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\begin{align*}
u_1 \times u_2 &= \{\alpha_1 x_1 + (1-\alpha_1)[-\frac{1}{2}\max\{x_2-x_1, 0\}]\} \times \{\alpha_2 x_2 + (1-\alpha_2)[-\frac{1}{2}\max\{x_1-x_2, 0\}]\},
\end{align*}
\]

**Proposition 1.** The equal income distribution, \(x_1 = x_2 = \frac{1}{2}\), is the unique maximizer of the Nash product \(u_1 \times u_2\).

**Proof.** The Nash product is a continuous function and hence has a maximizer in the unit simplex, say \((x_1, x_2)\). Moreover, it is easy to see that \(x_1 + x_2 = 1\).\(^4\) Therefore, we can set \(x \equiv x_1\) and \(x_2 \equiv 1 - x\). I will now prove that \(x \leq \frac{1}{2}\). Namely, independent of the weights \((\alpha_1, \alpha_2)\), agent 1 cannot obtain more than half the pie. Assume by contradiction that \(x > \frac{1}{2}\). In this case the Nash product is given by \(\alpha_1 x [\alpha_2 (1-x) - (1-\alpha_2)(x-\frac{1}{2})] = \alpha_1 x \left(\frac{1+\alpha_2}{2} - x\right)\). Therefore, \(x\) needs to maximize \(f(t) \equiv t \frac{1+\alpha_2}{2} - t^2\). Therefore, \(x \leq \frac{1}{2}\), because \(f'(t) = \frac{1+\alpha_2}{2} - 2t < 0\) on \((\frac{1}{2}, 1]\). The aforementioned arguments do not depend on the weights \((\alpha_1, \alpha_2)\), which means that agent 2, too, obtains at most half the pie: \(1 - x \leq \frac{1}{2}\). Therefore, \(x_1 = x_2 = \frac{1}{2}\). \(\square\)

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\(^3\)See, e.g., Binmore, Rubinstein, and Wolinsky (1986).

\(^4\)Let \(\epsilon \equiv 1 - x_1 - x_2\). If \(\epsilon > 0\) we can increase each \(x_i\) by \(\frac{\epsilon}{2}\), which will increase the value of the Nash product. To see this, note that since \(x_1 = x_2 = \frac{1}{2}\) is feasible, the value of the Nash product is strictly positive at the optimum. Also, it cannot be that both terms in the curly brackets are negative, because this would imply that \(x_2 > x_1\) and \(x_1 > x_2\). Therefore both are positive, and the aforementioned increment of \(\frac{\epsilon}{2}\) will increase either one.
3 An alternative relative deprivation measure

The relative deprivation measure $RD_i$ can be shown to equal the fraction of the individuals in the population whose incomes are higher than the income of individual $i$ times their mean excess income.\(^5\) Consider the alternative where the measure is simply taken to be the excess income, namely $\tilde{RD}_i \equiv \max\{x_j - x_i, 0\}$. Let the corresponding utility function be $\tilde{u}_i(x_1, x_2) = \alpha_i x_i + (1 - \alpha_i)\tilde{RD}_i(x_1, x_2)$. Stark et al. (2012) showed that their result is robust to this specification; that is, the equal income distribution is the unique maximizer of the sum $\tilde{u}_1 + \tilde{u}_2$. The following result shows that the same is true for the Nash solution.

**Proposition 2.** The equal income distribution, $x_1 = x_2 = \frac{1}{2}$, is the unique maximizer of the Nash product $\tilde{u}_1 \times \tilde{u}_2$.

**Proof.** Clearly a maximizer exists, and by the argument from the proof of Proposition 1 it is optimal to use the entire pie. Therefore, denote by $x$ and $1 - x$ the shares of agent 1 and agent 2, respectively. Suppose that $x > \frac{1}{2}$. The Nash product in this case is $\alpha_1 x[\alpha_2(1 - x) - (1 - \alpha_2)(2x - 1)] = \alpha_1 x[\alpha_2 x - (2 - \alpha_2)]$. Therefore, $x$ needs to maximize $g(t) \equiv t^2(\alpha_2 - 2) + t$. However, note that $g' = 0$ on $(\frac{1}{2}, 1]: 2t(\alpha_2 - 2) + 1 < 0$ if and only if $1 < 2t(2 - \alpha_2)$, which is true, because $\alpha_2 \in (0, 1)$. The possibility $x < \frac{1}{2}$ is similarly ruled out. \(\square\)

4 The Kalai-Smorodinsky solution

When there is disregard to relative deprivation (the relative deprivation measure is identically zero), it is trivial that the Nash solution to the division problem is $x_1 = x_2 = \frac{1}{2}$: this follows from *Pareto optimality*, *symmetry*, and *scale invariance*.\(^6\) One may therefore wonder whether Propositions 1 and 2 follow from the fact that

\(^5\)See Stark (2010).

\(^6\)Let $(S, d)$ be a generic bargaining problem and let $\mu$ be a bargaining solution. The solution $\mu$ is *Pareto efficient* if $\mu(S, d)$ is in $S$’s Pareto frontier, it is *symmetric* if $\mu_1(S, d) = \mu_2(S, d)$ whenever
the Nash solution satisfies these properties. This is not so: the Kalai-Smorodinsky solution (due to Kalai and Smorodinsky (1975)), for example, also satisfies these properties, but does not recommend equal sharing in our problem, except for the case that $\alpha_1 = \alpha_2$.

Formally, given a bargaining problem $(S, d)$, the ideal point of $(S, d)$, denoted by $a(S, d)$, is defined by $a_i(S, d) \equiv \max\{x_i : x \in S, x_j \geq d_j\}$. The Kalai-Smorodinsky solution to $(S, d)$ is the maximal point $s \in S$ such that $s_2 - d_2 s_1 - d_1 = a_2(S, d) - d_2$. Alongside Nash’s solution, this is the literature’s most prominent scale invariant solution.7

The derivation of the Kalai-Smorodinsky solution-point in the problem where the relative deprivation is given by $RD_i$ is as follows. The individually rational part of the feasible set consists of all the points “in between” the origin and the Pareto frontier. That frontier, in turn, is obtained by calculating the utility pairs that correspond to all non-wasteful allocations, $\{(x_1, x_2) = (x, 1-x) : x \in [0,1]\}$. Consider $x < \frac{1}{2}$. The corresponding utilities are $u_1 = \alpha_1 x - (1 - \alpha_1)(\frac{1}{2} - x) = -\frac{1}{2} + x + \frac{\alpha_1}{2}$ and $u_2 = \alpha_2(1-x)$. The ideal payoff for player 2 is obtained when setting $u_1 = 0$, or $x = \frac{1-\alpha_1}{2}$, and the corresponding point in the feasible set is $(0, \frac{\alpha_2(1+\alpha_1)}{2})$. Considering the symmetry case (agent 1 receives more than half the pie), we see that the ideal point is $(\frac{\alpha_1(1+\alpha_2)}{2}, \frac{\alpha_2(1+\alpha_1)}{2})$ which implies that the Kalai-Smorodinsky solution dictates equal sharing of the pie if, and only if, $\alpha_1 = \alpha_2$.8

In the context of an arbitration model, Kalai and Rosenthal (1978) considered a variant of the Kalai-Smorodinsky solution, which is described as follows: the solution point to $(S, d)$ is the maximal $s \in S$ such that $\frac{s_2 - d_2}{s_1 - d_1} = \frac{A_2(S, d) - d_2}{A_1(S, d) - d_1}$, where $A(S, d)$, the unrestricted ideal point of $(S, d)$, is defined by $A_i(S, d) \equiv \max\{x_i : x \in S\}$. $S$ is symmetric around the 45°-line and $d$ is on that line, and it is scale invariant if $T(\mu(S, d)) = \mu(T(S), T(d))$ for every positive affine transformation $T$.

7See Thomson (1994).

8The ratio of the ideal payoffs is $\frac{\alpha_2(1+\alpha_1)}{\alpha_1(1+\alpha_2)}$, hence $\frac{\alpha_2(1+\alpha_1)}{\alpha_1(1+\alpha_2)} = \frac{\alpha_2}{\alpha_1}$ (the RHS is the payoff-ratio under equal sharing) if and only if $\alpha_1 = \alpha_2$.

9It is easy to check that the same conclusion is reached in the case that relative deprivation is given by $RD_i$. 
Interestingly, as opposed to the Kalai-Smorodinsky solution, this solution does dictate equal sharing independent of \((\alpha_1, \alpha_2) \in (0, 1)^2\).

**Proposition 3.** Consider the division problem with the utility functions \((u_1, u_2)\). The Kalai-Rosenthal solution-point to this problem, \(k\), satisfies \(k_1 = k_2 = \frac{1}{2}\).

**Proof.** Note that \(A(S, d) = (\alpha_1, \alpha_2)\) and that \(k_2 = 1 - k_1\). Suppose that \(k_1 > \frac{1}{2}\). In this case, \(\frac{\frac{1}{2} + \alpha_2 - k_1}{\alpha_1 k_1} = \frac{\alpha_2}{\alpha_1}\), hence \(k_1 = \frac{1}{2}\), a contradiction; \(k_1 < \frac{1}{2}\) is ruled out similarly. \(\square\)

**Proposition 4.** Consider the division problem with the utility functions \((\tilde{u}_1, \tilde{u}_2)\). The Kalai-Rosenthal solution-point to this problem, \(\tilde{k}\), satisfies \(\tilde{k}_1 = \tilde{k}_2 = \frac{1}{2}\).

**Proof.** By efficiency, \(\tilde{k}_2 = 1 - \tilde{k}_1\). Consider first \(\tilde{k}_1 > \frac{1}{2}\). In this case, the following equation must hold: \(\frac{k_1(\alpha_2 - 2) + 1}{\alpha_1 k_1} = \frac{\alpha_2}{\alpha_1}\), implying \(\tilde{k}_1 = \frac{1}{2}\), a contradiction. The possibility \(\tilde{k}_1 < \frac{1}{2}\) is ruled out similarly. \(\square\)

**References**


