In this Appendix I establish the independence of the axioms of Theorem 1, I discuss the extension to an arbitrary (finite) number of players, and I prove that the Nash solution, $N$, does not satisfy the axiom RSA.

1 Independence of the axioms

The midpoint solution, $m(S,d) \equiv \frac{1}{2}d + \frac{1}{2}a(S,d)$, satisfies all of the axioms from Theorem 1 but PO.\footnote{Consider the following modification of $m(S,d)$: $m^*(S,d) \equiv m(S,d) + (e,e)$, where $e$ is the maximal number such that the aforementioned expression is in $S$. It is easy to check that this solution satisfies WPO, MD, IM, and RSA; it shows that PO cannot be weakened to WPO without rendering the conclusion of Theorem 1 false.} The equal loss solution, $EL(S,d) \equiv a(S,d) - (l,l)$, where $l$ is the minimal number such that the aforementioned expression is in $S$, satisfies all of them but MD (this solution is due to Chun (1988)). The following solution satisfies all the axioms but RSA. Given an arbitrary $(S,d)$, let us denote, for short, its ideal point and midpoint by $a$ and $m$, respectively. Consider the following piecewise linear monotone path solution: it assigns to each $(S,d)$ the point $P(S) \cap \{[m; (\frac{1}{2}a_1 + \frac{1}{2}m_1, m_2)] \cup [(\frac{1}{2}a_1 + \frac{1}{2}m_1, m_2)] \cup [\frac{1}{2}a_1 + \frac{1}{2}m_1, m_2])$.}
$\{m_1, m_2; a]\}$. It is easy to see that this solution satisfies PO, MD, and IM.

As an example of a solution that satisfies all the axioms but IM, consider the following bargaining solution, the *Perles-Maschler solution*, $PM$. This solution is defined on $B_0$: the class of problems $(S, d) \in B$ where $d = 0$, $S = S_d$, and $WP(S) = P(S)$. Given $(S, d)$ as above, $PM(S, d)$ is the unique point $u \in P(S)$ that satisfies:

$$\int_{(0, a_2)}^{u} \sqrt{-d} \, dx \, dy = \int_{u}^{(a_1, 0)} \sqrt{-d} \, dx \, dy,$$

where the integrals are taken along the arcs of $\partial S = P(S)$. It is well-known that this solution satisfies PO and RSA (in fact: SA and SA*), and that it violates IM. It also satisfies MD.

2 Extension to $n \geq 3$ players

As mentioned in the paper, the model and the main result (Theorem 1) extend straightforwardly to the case of $n \geq 3$ players. However, some remarks regarding the independence of the axioms are in place. The midpoint solution and the piecewise linear monotone path solution from above have counterparts in the multi-person case. The equal-loss solution, on the other hand, may fail to exist when there are more than two players. Nevertheless, it is well-defined in the ($n$-person) model in which the assumption of compact feasible sets is replaced by unboundedness from below.

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2The choice $d \equiv 0$ is a mere normalization; equivalently, one can consider the collection of all $(S, d)$ with a common $d$ and where $S = S_d$, which is simply a $d$-translation of the Perles-Maschler setting.

3The solution $PM$ can be defined also for problems for which $WP(S) \neq P(S)$, but then the expression (1) needs to be amended in order to account for the possibility that the Pareto boundary contains a segment parallel to an axis. This is only a technically that I will ignore for the sake of the ease of presentation.

or free disposal. Finally, the existence of an $n$-person solution that satisfies all the axioms but IM is also nontrivial. Here is an example of a rich domain of 3-person problems on which such a solution exists.

Given a 3-person bargaining problem $(S, d)$, define the set $X(S, d)$ as follows:

$$X((S, d)) \equiv \{(a, b) : (a, b, m_3(S, d)) \in S\}.$$  

Consider the domain of smooth 3-person problems: those 3-person $(S, d)$ such that $WP(S) = P(S)$, and where $P(S)$ does not contain hyperplanes; that is, for all distinct $x, y \in P(S)$ and $\alpha \in (0, 1)$, the point $\alpha x + (1 - \alpha)y$ is not in $P(S)$. Define the solution $\mu^*$ on this domain by:

$$\mu^*(S, d) \equiv (PM_1(X((S, d)), (d_1, d_2)), PM_2(X((S, d)), (d_1, d_2)), m_3(S, d)).$$

It is easy to see that this solution satisfies PO, RSA, and MD.

3 $N$ does not satisfy RSA

Let $S = \text{conv hull}\{0, (1, 0), (0, 2)\}$ and let $T \equiv \text{conv hull}(S \cup \{(1, 1 + \epsilon)\})$, for some small $\epsilon > 0$. Let $Q \equiv \frac{1}{1+\epsilon}S + \frac{1}{1+\epsilon}T$. We have that $N(S, 0) = (\frac{1}{2}, 1)$, $N(T, 0) = (1, 1+\epsilon)$, and $N(Q, 0) = (1, 1)$. The requirement of RSA fails for player 2.

References


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5In such a model, RSA needs to be amended so that “$S = S_d$” (“$T = T_d$”) is replaced by “$S =$ comprehensive hull of $S_d$” (“$T =$ comprehensive hull of $T_d$”).