Interpersonal Utility Comparisons in the Bargaining Problem

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Abstract

Competing schools of thought offer alternative interpretations to Nash’s bargaining model: according to one, the utilities in that model are not interpersonally comparable, while according to the other they certainly are. The former position is assumed by the Nash and Kalai-Smorodinsky solutions, and the latter is assumed by the egalitarian, utilitarian, and equal-loss solutions. In this paper, I describe a certain form of equivalence between the set consisting of the former solutions and the set consisting of the latter. This equivalence is the result of an attempt to bridge the gap between the aforementioned views; utilizing this equivalence, I derive a new characterization of the Nash solution. As an intermediate result, I also derive a new characterization of the Kalai-Smorodinsky solution.

Keywords: Bargaining; interpersonal utility comparisons. JEL Codes: D63; D71

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1 Introduction

The Nash bargaining problem (due to Nash (1950)) is defined as a pair \((S, d)\). The set \(S\), the feasible set, consists of all the utility vectors that the players can achieve via unanimous agreement, and \(d \in S\), the disagreement point, specifies their utilities in case that no such agreement is reached—player \(i\) receives the utility payoff \(d_i\) in this event.

The utilities in the bargaining problem are v-N.M utilities, but there are two ways to interpret them. According to one, a player’s utility in all problems is measured on the same scale, e.g., in terms of dollars. According to another view, utilities in different problems are not necessarily measured on the same scale. Both interpretations are consistent with expected utility: under either one, any pair of utility allocations \(x, y \in S\) describe v-N.M utility payoffs, and the mixture \(z = \alpha x + (1 - \alpha) y\) describes the expected utilities from the lottery that yields \(x\) with probability \(\alpha\) and \(y\) otherwise.\(^1\) The numbers \(x_i, y_i\), and \(z_i\) are measured on the same scale. However, under the first interpretation we know what is that scale, while under the second we don’t. Consequently, the former allows for interpersonal utility comparisons, the latter does not.\(^2\) I will therefore call the former approach the interpersonal approach (IPA) and call the latter approach the interpersonal free approach (IPFA).

A bargaining solution picks a unique agreement—a point in the feasible set—for every problem. Suppose, as an illustration, that there is an impartial

\(^1\)It is assumed that the feasible set of every problem is convex; a precise and formal description of the model will be given in Section 2.

\(^2\)If we know player \(i\)'s problem-independent utility scale for each \(i\), and we additionally know the “conversion rate” between these scales, then we can compare utilities, no matter the problem. There is a philosophical concern of whether such a comparison is, in principle, meaningful. It will not bother us in this paper.
arbitrator who solves bargaining problems on behalf of the players. A solution, in this case, is a description of the arbitrator’s recommendations.

The major implication of IPA is that it makes a significant assumption on this arbitrator: it assumes that he knows the common utility scale despite the fact that, formally speaking, this scale is not a part of the mathematical model. In other words, it assumes that the arbitrator is equipped with knowledge from outside the utility space. This assumption may not seem too strong in applied models. For example, it goes hand in hand with the following view. Suppose that all bargaining problems take place in a two-commodity world, in which preferences over one of the goods, “money,” are linear, and bargaining is always about the division of the non-monetary good. In this world, each player $i$ has preferences over pairs of the form $(y_i, t_i)$, where $t_i$ is his wealth and $y_i$ is the amount of the non-money good he is allocated. These preferences are not only v-N.M, but, moreover, admit the following utility representation: $U_i(y_i, t_i) = v_i(y_i) + t_i$. The utilities in the bargaining problem, then, assume a concrete interpretation—willingness to pay. That is, with $v_i(0) = 0$, player $i$ is willing to pay up to $x_i$ for the agreement $(x_1, \cdots, x_n) \in S$.

However, without outside-the-utility-space information the aforementioned picture has no room, and IPFA prevails. This formality, to be sure, has its merits; however, it also has a cost: it makes the consideration of fairness and efficiency in bargaining quite difficult.

Concepts related to fairness and efficiency are typically meaningful only under IPA. Think of the utilitarian criterion of maximizing the sum of the players’ utilities and the egalitarian criterion of equating these utilities. What sense does it make to add “apples to oranges”? Likewise, what reason is there to equate meaningless numbers? It therefore seems that adopting IPFA, as
is widely done in the literature,\textsuperscript{3} makes it impossible, from the very onset, to consider basic ideas of fairness and efficiency, at least in their traditional formulation.

The point of this paper is to argue the opposite. I offer a middleground which combines IPA and IPFA. In this middleground, there is room for both scale invariance,\textsuperscript{4} and, at the same time, there is room for operations such as summing up and equating payoffs. Taking this approach, I derive a new characterization of the 2-person Nash bargaining solution. As an intermediate result, I also derive a new characterization of the 2-person Kalai-Smorodinsky bargaining solution (due to Kalai and Smorodinsky (1975)).

My idea is this. In the existing literature, under either IPA or IPFA, the bargaining model assumes a fixed set of players. Instead, I offer the following alternative view: imagine that each problem $B = (S,d)$ corresponds to a possibly different set of players, $(1_B, 2_B, \cdots, n_B) \equiv I(B)$. When each problem corresponds to potentially different players, scale invariance does not assume its standard interpretation anymore—it becomes a requirement that regards comparisons of different sets of players. Under this interpretation of the model, interpersonal utility comparisons make sense. After all, if the problem $B$ is unique to the players in $I(B)$, we may very well assume that, to begin with, it is given in the scales that capture the right interpersonal comparisons among them. It is therefore meaningful to consider the following solutions to $B = (S,d)$. The utilitarian solution, $U(S,d)$, the egalitarian solution, $E(S,d)$, and the equal-loss solution, $EL(S,d)$. The first is defined to be any selection

\textsuperscript{3}See Thomson (1994) for a comprehensive survey of the literature.

\textsuperscript{4}Namely, for the requirement that for every positive affine transformation $\lambda$, and every problem $(S,d)$, the solution $\mu$ satisfies $\mu(\lambda \circ S, \lambda \circ d) = \lambda \circ \mu(S,d)$.
from $U(S, d) \equiv \arg\max_{S_d} \sum x_i$, where $S_d \equiv \{x \in S | x \geq d\},^5$ the second is
given by $d + \epsilon \cdot 1,^6$ where $\epsilon$ is the maximal number such that the aforemen-
tioned expression is in $S$, and the third picks the highest point $y \in S$ such
that $a_i - y_i = a_j - y_j$ for all $i$ and $j$, where $a_i = a_i(S, d) \equiv \max\{x_i | x \in S_d\}.^7,^8$

The solution $U$ represents the classical idea of utilitarianism, which dates
back to Jeremy Bentham. In an environment with a common transferable
utility unit, maximization of the sum of the individuals' utilities is equivalent
to Pareto efficiency. The other solutions, $E$ and $EL$, represent two notions of
fairness: equality of gain and equality of sacrifice. All of these solutions are
sensible under IPA. However, since they typically yield different recommenda-
tions, they present bargaining theory with substantial difficulties: first, in
solving bargaining situations, one needs to compromise on either fairness or
efficiency; second, to begin with, it is not obvious what is the meaning of a
“fair outcome,” since there are (at least) two reasonable notions of fairness.

Underlying these difficulties is a more general problem. Suppose that the
arbiter considers $\{s_1, \cdots, s^K\}$ as legitimate candidate-solutions. If each so-
lution has its merits, in terms of the axiomatizations it enjoys, the philosoph-
ical ideas it expresses, or otherwise, it is not clear which one he should chose.
A conservative first step would be to demand that payoffs never fall below
the minimum of the recommendations of these solutions; namely, that in each
problem $(S, d)$ each player $i$ would receive at least $\min\{s_1^i(S, d), \cdots, s^K_i(S, d)\}$.
This is a notion of insurance that guarantees that payoffs will never fall short
of a certain bound—a bound which is itself a function of the appealing, but
jointly-inconsistent, solutions $(s^K)_{k=1}^K$. Alternatively, one can think of the fol-

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^5 Vector inequalities: $x_R y$ if and only if $x_R y_i$ for all $i$, for both $R \in \{\geq, >\}$.
^6 $1 = (1, \cdots, 1)$; similarly, $0 = (0, \cdots, 0)$.
^7 $EL(S, d)$ is well-defined for $n = 2$ but may fail to exist when $n \geq 3$.
^8 $E$ was first characterized by Kalai (1977); $EL$ was first characterized by Chun (1988).
ollowing noncooperative justification for this idea: the players are seating at
the negotiations table, where the agreement \( x \in S \) is up for consideration. If
\( x_i \geq \min\{s_i^1(S,d), \cdots, s_i^K(S,d)\} \) for each player \( i \), then the candidate agree-
ment \( x \) is robust in the following sense: if some player \( j \) complains that \( x_j \) is
too low, then the other players (or the arbitrator) can reject his complain on
the basis that “\( s^k \) is a legitimate solution, and under \( x \) you receive a payoff
which is greater than the one you would have obtained under \( s^k \).”

Going back to our original concerns, this rationale leads me to consider the
following requirement: to demand that for each bargaining problem \((S,d)\) each
player \( i \) will receive a payoff which is at least \( \min\{U_i(S,d), E_i(S,d), EL_i(S,d)\} \).\(^9\)
I say that such a solution satisfies restricted interpersonal comparisons.
For 2-person problems, a continuous scale-invariant solution exhibits this prop-
erty if and only if the following is true: for every problem \((S,d)\), each player \( i \)
receives a payoff which is at least as large as \( \min\{N_i(S,d), KS_i(S,d)\} \), where
\( N \) is the Nash bargaining solution and \( KS \) is the Kalai-Smorodinsky solution.
Thus, this paper uncovers a nontrivial link between the major IPA-solutions
and the major IPFA-solutions.

I view restricted interpersonal comparisons as a simple (and rather weak)
way of entertaining the IPA-philosophy into the bargaining model. A more
formal and comprehensive treatment of how to go about this task is beyond
the scope of the present paper. The general principle of constructing the lower
bound \( \min\{s_i^1(S,d), \cdots, s_i^K(S,d)\} \) can be applied, of course, to other lists of
solutions. In a previous paper, I considered the case \( \{s^1, s^2\} = \{E, U\} \). Based
on the associated axiom, I characterized the 2-person Nash solution. In Sec-
tion 4 below I show that considering the bound that corresponds to \( \{s^1, s^2\} =

\(^9\)To be more precise, the requirement is that there exists a selection \( U \) from \( U \) such that
the above is satisfied. See Section 4 for the formal definition.
\{E, EL\} leads to a characterization of the 2-person Kalai-Smorodinsky solution.\(^\text{10}\)

The rest of the paper is organized as follows. Section 2 lays down the model. Section 3 discusses its varying-set-of-players interpretation. Section 4 discusses the background for the main result, which comprises of two parts: survey of relevant results from previous literature and the aforementioned new characterization of KS. Section 5 contains the main result. The results of this paper hold for 2-person bargaining, but cannot be extended to multi-person bargaining; this point is discussed briefly in Section 6.\(^\text{11}\) Section 7 concludes.

2 Model

An \(n\)-person bargaining problem (a problem, for short) is a pair \((S, d)\) such that \(S \subset \mathbb{R}^n\) is closed and convex, \(d \in S\) is such that \(S_d \equiv \{ x \in S | x \geq d \}\) is bounded and contains a point \(x\) such that \(x > d\), and \(S\) is \(d\)-comprehensive; that is, for all \(x \in S\): \(d \leq y \leq x \Rightarrow y \in S\). The collection of all these pairs \((S, d)\) is denoted \(\mathcal{B}_n\). A problem \((S, d)\) is smooth if the Pareto frontier of its feasible set does not contain segments; that is, if for distinct \(x, y \in P(S)\) and \(\alpha \in (0, 1)\) the point \(\alpha x + (1 - \alpha)y\) is not in \(P(S)\), where \(P(S) \equiv \{ x \in S | y \geq x \& y \neq x \Rightarrow y /\in S \}\).

A solution is any function \(\mu: \mathcal{B}_n \rightarrow \mathbb{R}^n\) that satisfies \(\mu(S, d) \in S\) for all \((S, d) \in \mathcal{B}_n\). The Nash solution (due to Nash (1950)), \(N\), is the unique maximizer of \(\Pi_{i=1}^n(x_i - d_i)\) over \(x \in S_d\). The Kalai-Smorodinsky solution (due to Kalai and Smorodinsky (1975)), \(KS\), is given by \((1 - \theta)d + \theta a(S, d)\), where \(\theta\) is

\(^{10}\)The bound that corresponds to \(\{s^1, s^2\} = \{U, EL\}\), on the other hand, is inconsistent with scale invariance.

\(^{11}\)“2 players” versus “multiple players” should be read simply as “2 dimensions” versus “\(n\) dimensions.” After all, it is a central idea of this paper that there may be many “real” players even if the dimension of the utility space is 2.
the maximal number such that the aforementioned expression is in \( S \), where 
\[ a_i(S, d) \equiv \max\{x_i | x \in S_d\}. \]
Let \( E \) and \( EL \) denote the egalitarian and equal-loss solutions, respectively, and let \( U \) denote the utilitarian correspondence (see Section 1 above for their definitions).

The following axioms will be of interest in the sequel. In their definitions, 
\( (S, d) \in \mathcal{B}_n \) is an arbitrary problem and \( \{ (S_k, d) \}_k \subset \mathcal{B}_n \) is an arbitrary sequence of problems with a common disagreement point.

**Pareto Optimality** (PO): \( \mu(S, d) \in P(S) \).

**Disagreement Convexity** (D.VEX): 
\[ \mu(S, \alpha d + (1 - \alpha)\mu(S, d)) = \mu(S, d) \] for all \( \alpha \in (0, 1] \).

**Midpoint Domination** (MD): 
\[ \mu(S, d) \geq \frac{1}{n}\sum_{i=1}^{n}(a_i(S, d), d_i). \]

**Scale Invariance** (SINV): 
\( \lambda \circ \mu(S, d) = \mu(\lambda \circ S, \lambda \circ d) \) for every positive affine transformation \( \lambda : \mathbb{R}^n \to \mathbb{R}^n \).

**Continuity** (CONT): If \( \{S_k\} \) converges to \( S \) in the Hausdorff metric, then \( \{\mu(S_k, d)\} \) converges to \( \mu(S, d) \).

PO is obvious. D.VEX says that a movement of the disagreement point in the direction of the agreement should not change the agreement and MD says

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\(^{12}\)The point \( a(S, d) \) is called the **ideal point** of the problem \( (S, d) \).

\(^{13}\)Note that if \( (S, d) \) is a smooth problem then \( U(S, d) \) is a singleton; in this case, we can unambiguously talk about the utilitarian solution to \( (S, d) \).

\(^{14}\)A function \( \lambda : \mathbb{R}^n \to \mathbb{R}^n \) is a positive affine transformation if \( \lambda \circ (x_1, \ldots, x_n) \equiv (\lambda_1 x_1, \ldots, \lambda_n x_n) + t \) for some numbers \( \lambda_i > 0 \) and a vector \( t \in \mathbb{R}^n \) (a translation).
that the agreement should dominate “randomized dictatorship.” These ideas are well-known and thoroughly discussed in the literature.\footnote{See, e.g., Thomson (1994).}

### 3 An alternative interpretation of the model

Recall the varying-set-of-players interpretation: imagine that every problem $B = (S, d)$ corresponds to a potentially different set of players, $I(B)$. For every $B$ and $I(B)$, the axioms PO and MD assume their standard interpretations. Assuming, for simplicity, that $I(B)$ only depends on $S$, D.VEX also assumes its standard interpretation. SINV and CONT are different, since they refer to multiple problems: the former refers to all the positive affine transformations of a given problem and the latter refers to a sequence of problems. Under the varying-set-of-players interpretation, therefore, they refer to problems played by (possibly) different bargainers. The interpretation of CONT is straightforward: if two groups of bargainers are “similar,” then the agreements that they reach should also be. The interpretation of SINV, on the other hand, is less obvious.

To get a better understanding of what SINV means in the varying-set-of-players setting, the following observation is useful. Note that for two distinct problems, $B$ and $B'$, the corresponding sets of bargainers, $I(B)$ and $I(B')$, can be the same individuals but in different points in time, under different circumstances, and so on. Then, comparisons of different problems can assume a broad spectrum of applications, since the comparison is (at the level of interpretations) between two sets of circumstances under which bargaining takes place. The following 2-person examples illustrate this point; the interpretation of SINV will be explained following these examples.
Example 1. In the problem $B = (S, d)$ the two players need to divide a dollar; each player’s utility is linear in money and the disagreement point is $d = (0, 0)$. The problem $B' = (\lambda \circ S, \lambda \circ d)$ is identical, except that player 2 is facing the following risk: after bargaining ends and he goes home, he may lose what he got with probability $p$. In this case, $\lambda(x, y) = (x, (1 - p)y)$.

Example 2. $B = (S, d)$ is as above and $B'$ is “wait one period and then play $B$.” Here, $\lambda(x, y) = (\delta_1 x, \delta_2 y)$, where $\delta_i$ is player $i$’s one-period discount factor.

In the 2-person case, under the assumption of PO, SINV can be interpreted as an independence principle: it restricts the extend to which the identity of your partner, or the circumstances to which he is subjected, can affect your payoff. So, for instance, in Example 1, your payoff should be independent of the probability that your opponent will lose his share; in Example 2, your payoff should be independent of your partner’s time preference.

4 Background for the main result

Under the varying-set-of-players interpretation, SINV does not exclude the possibility of interpersonal comparisons of utility. In particular, the arbitrator may have fairness and efficiency in mind as arbitration-guidelines; such an arbitrator may find both the egalitarian solution, $E$, as well as the utilitarian solution, $U$, appealing. However, since these solutions typically lead to different recommendations, he will need to find a method to produce some sort of compromise between them. One simple such method is to demand that for every problem $(S, d)$, no player will receive less than $\min\{E_i(S, d), U_i(S, d)\}$. 

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In Rachmilevitch (2011), I showed that this requirement, when combined with SINV, pins down the Nash solution in the 2-person case. Formally,

**Proposition 1.** (Rachmilevitch 2011) Let \( \mu \) be a solution on \( B_2 \) that satisfies SINV and CONT. Then, there exists a selection \( U \) from \( U \) such that \( \mu_i(S, d) \geq \min\{U_i(S, d), E_i(S, d)\} \) for all \( (S, d) \in B_2 \) and \( i \in \{1, 2\} \) if and only if \( \mu = N \).

This result cannot be extended to the multi-person case; in fact, this impossibility remains even in the absence of CONT. Formally,

**Proposition 2.** (Rachmilevitch 2011) Let \( \mu \) be a solution on \( B_n \) that satisfies SINV. Then, there exists a selection \( U \) from \( U \) such that \( \mu_i(S, d) \geq \min\{U_i(S, d), E_i(S, d)\} \) for all \( (S, d) \in B_n \) and \( 1 \leq i \leq n \) if and only if \( n = 2 \).

The idea of restricted interpersonal comparisons is obtained by weakening the lower bound, by replacing it with \( \min\{E_i(S, d), U_i(S, d), EL_i(S, d)\} \). Formally,

**Restricted Interpersonal Comparisons (RIC):** A solution \( \mu \) satisfies restricted interpersonal comparisons if there exists a selection \( U(S, d) \in U(S, d) \) such that

\[
\mu_i(S, d) \geq \min\{U_i(S, d), E_i(S, d), EL_i(S, d)\} \quad \forall i.
\]

Of course, instead weakening the bound, one can consider, in a similar spirit, alternative bounds. For example, one can impose on the solution the requirement that payoffs should never fall below \( \min\{E_i(S, d), EL_i(S, d)\} \). This variation, it turns out, leads to a characterization of the Kalai-Smorodinsky solution.

**Proposition 3.** Let \( \mu \) be a solution on \( B_2 \) that satisfies SINV. Then \( \mu_i(S, d) \geq \min\{E_i(S, d), EL_i(S, d)\} \) for all \( (S, d) \in B_2 \) and \( i \in \{1, 2\} \) if and only if \( \mu = KS \).
Proof. I start by proving that $KS_i(S,d) \geq \min\{E_i(S,d), EL_i(S,d)\}$ for all $(S,d) \in B_2$ and $i \in \{1,2\}$. Let $(S,d) \in B_2$. Let $a \equiv a(S,d)$, let $k \equiv KS(S,d)$, and let $x \equiv EL(S,d)$. Wlog, suppose that $d = (0,0)$. If $a_1 = a_2$ then $k = E(S,d) = EL(S,d)$ and we are done. Suppose then, wlog, that $a_1 > a_2$. Obviously, we can also assume, wlog, that $a_1 = 1$. In this case $k_1 > E_1(S,d); k_2 \geq x_2$. I will prove that $k_2 \geq x_2$. Assume by contradiction that $k_2 < x_2$. Since $\frac{k_2}{k_1} = \frac{a_2}{a_1} = a_2$ we have $k_2 = a_2k_1$. Therefore $a_2k_1 < x_2$. By the definition of $EL$, $a_1 - x_1 = a_2 - x_2$, hence $x_2 = a_2 + x_1 - a_1 = a_2 + x_1 - 1$. Therefore $a_2 + x_1 - 1 > a_2k_1 \Rightarrow a_2(1-k_1) > 1-x_1$, and since $a_2 < a_1 = 1$, $1-k_1 > 1-x_1$, or $x_1 > k_1$. We obtain that $x_i > k_i$ for both $i \in \{1,2\}$, in contradiction to the Pareto optimality of $KS$.

I now turn to prove uniqueness. Let $\mu$ be a scale-invariant solution that satisfies $\mu_i(S,d) \geq \min\{E_i(S,d), EL_i(S,d)\}$ for all $(S,d) \in B_2$ and $i \in \{1,2\}$. Let $(S,d) \in B_2$. Let $\lambda$ be the linear transformation given by $\lambda \circ x = \left(\frac{a_2(S)}{a_1(S)}x_1, x_2\right)$. Note that $E(T,d) = EL(T,d) = KS(T,d)$, where $T \equiv \lambda \circ S$. Since $\mu_i(S,d) \geq \min\{E_i(S,d), EL_i(S,d)\}$ for all $(S,d) \in B_2$ and $i \in \{1,2\}$, $\mu(T,d) \geq KS(T,d)$ and therefore $\mu(T,d) = KS(T,d)$. Since both $\mu$ and $KS$ are scale invariant, $\mu(S,d) = KS(S,d) = \lambda^{-1} \circ E(T,d)$.

As was the case with Proposition 1, Proposition 3, too, cannot be extended to multi-person bargaining. First of all, $EL$ may fail to exist when there are more than two players. Moreover, even if Proposition 3 is amended as to demand $\mu_i(S,d) \geq \min\{E_i(S,d), EL_i(S,d)\}$ for all $i$ only for problems for which $EL(S,d)$ exists, the impossibility remains: one can easily show that if there is a multi-person scale invariant solution that satisfies the aforementioned requirement, then it must be $KS$; however, it is not hard to find examples where the multi-person $KS$ violates this requirement.
Finally, one more result that will be useful is the sequel is the following, due to de Clippel (2007).

**Proposition 4.** (de Clippel 2007) \( N \) is the unique solution on \( \mathcal{B}_2 \) that satisfies D.VEX and MD.\(^{16}\)

## 5 The main result

Building on the results from the previous section, the result below describes a formal connection between \( N, KS \), and \( RIC \).

**Proposition 5.** Let \( \mu \) be solution on \( \mathcal{B}_2 \) that satisfies SINV and CONT. Then \( \mu \) satisfies RIC if and only if \( \mu_i(S,d) \geq \min\{N_i(S,d), KS_i(S,d)\} \) for all \( (S,d) \in \mathcal{B}_2 \) and \( i \in \{1, 2\} \).

**Proof.** Make the assumptions of the proposition. Assume first that \( \mu_i(S,d) \geq \min\{N_i(S,d), KS_i(S,d)\} \) for all \( (S,d) \in \mathcal{B}_2 \) and \( i \in \{1, 2\} \). Then, in view of Proposition 1 and Proposition 3, \( \mu \) satisfies RIC. Conversely, suppose that it satisfies RIC and let \( (S,d) \in \mathcal{B}_2 \). We need to show that for each player \( i \) the following holds: \( \mu_i(S,d) \geq \min\{N_i(S,d), KS_i(S,d)\} \). By CONT we may assume that \( (S,d) \) is smooth. Let \( \lambda \) be the positive affine transformation such that \( E(T,d') = U(T,d') = N(T,d') = x \), where \( T \equiv \lambda \circ S \) and \( d' \equiv \lambda \circ d \).\(^{17}\) Wlog, suppose that \( d = d' = 0 \); let \( y \equiv KS(T,0) \) and \( a \equiv a(T,0) \).

Suppose first that \( a_1 = a_2 \). In this case \( y = KS(T,0) = EL(T,0) = E(T,0) = U(T,0) = N(T,0) = x \), and since \( \mu \) satisfies RIC, \( \mu(T,0) \geq x \); since

\(^{16}\)de Clippel’s result is more general than that (it allows the solution to be multi-valued), but for our purpose the aforementioned version will suffice.

\(^{17}\)The existence of this positive affine transformation \( \lambda \) was first established by Harsanyi (1959). Shapley (1969) presents a related, and slightly more generally-formulated result.
Suppose, on the other hand, that \( a_1 \neq a_2 \); wlog, suppose that \( a_2 > a_1 \). In this case, \( y \) is to the north west of \( x \). Let \( z \equiv \mu(T,0) \). By SINV, it is enough to prove that \( z_i \geq \min\{N_i(T,0), KS_i(T,0)\} \) for both \( i \in \{1,2\} \). That is, that \( z_1 \geq y_1 \) and that \( z_2 \geq x_2 \). The second inequality follows directly from RIC, because \( EL(T,0) \) is to the north west of \( x \) (in fact, it is to the north west of \( y \)).

Assume, contrawise, that \( z_1 < y_1 \). Let \( \lambda' \) be the positive affine transformation \((A,B) \mapsto (A, a_1 A + a_2 B)\) and let \( Q \equiv \lambda' \circ T \). By SINV, \( \mu_1(Q,0) = z_1 < \min\{U_1(Q,0), E_1(Q,0), EL_1(Q,0)\} \), in contradiction to RIC.

We can now turn to the main result.

**Theorem 1.** \( N \) is the unique solution on \( B_2 \) that satisfies RIC, D.VEX, SINV, and CONT.

**Proof.** It is well-known that \( N \) satisfies D.VEX, SINV, and CONT, and it follows from Proposition 1 that it also satisfies RIC. Conversely, let \( \mu \) be an arbitrary solution on \( B_2 \) that satisfies the four axioms. By Proposition 5, it satisfies
\[
\mu_i(S,d) \geq \min\{N_i(S,d), KS_i(S,d)\}
\]
for all \((S,d) \in B_2\) and \( i \in \{1,2\} \). Therefore, since both \( N \) and \( KS \) satisfy MD, \( \mu \) satisfies MD. By Proposition 4, the combination of MD and D.VEX implies \( \mu = N \).

The axioms are independent. \( KS \) satisfies all of them but D.VEX, \( E \) satisfies all of them but SINV, and \( \mu(S,d) \equiv d \) satisfies all of them but RIC. I now turn to describe a solution that satisfies all the axioms but CONT.

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18 This argument implicitly relies on the smoothness of \( S \). In its absence, we cannot conclude that \( z_2 \geq x_2 \), because the selection \( U \) in terms of which RIC is defined may select a point to the south east of \( x \).

19 Note that \( \lambda' \circ y = EL(Q,0) = E(Q,0) \); the fact that the utilitarian solution must weakly move to the right when \( \lambda' \) is applied to \( T \) completes the proof.
Let $S_x \equiv \text{conv} \{ (0,0), (0,1), (x,1), (2x,0) \}$ and let $D_x \equiv \{ \beta \times (0,0) + (1-\beta) \times (x,1) | 0 \leq \beta \leq 1 \}$ be the diagonal that connects the origin to $(x,1)$. Define the following solution, $\mu^*$, as follows. For $(S,d)$ such that $S = S_x$ and $d$ is to the left of $D_x$, let $\mu^*(S,d) = (x,1)$. If $(S,d)$ is such that there exists a positive affine transformation $\lambda$ such that $(\lambda \circ S, \lambda \circ d)$ is a problem of the kind that was described in the previous sentence, let $\mu^*(S,d) = \lambda^{-1} \circ (x,1)$. For any other $(S,d)$, set $\mu^*(S,d) = N(S,d)$. Clearly $\mu^*$ satisfies D.VEX and SINV. As for RIC, we only need to establish it for the case where $S = S_x$ for some $x > 0$ and $d$ is to the left of $D_x$. If $x < 1$, then $\mu^*(S,d)$ is the unique utilitarian point in $S_d$. If, on the other hand, $x \geq 1$, then $\mu^*(S,d)$ is weakly to the right of $E(S,d)$.

6 Multiple players

For multi-person bargaining, i.e., for $n \geq 3$, RIC must be amended, because $EL$ may fail to exist in this case.\footnote{$EL$ is guaranteed to exist if the feasible set is unbounded from below; that is, if there is free disposal of utility. This paper, however, allows for compact feasible sets, hence existence is an issue. Moreover, even when $EL$ is guaranteed to exist (as under the additional assumption of free disposal), it presents an additional difficulty for multi-person bargaining: it is not individually rational: that is, $EL_i(S,d) < d_i$ is possible.} One way to try to remedy the problem is by strengthening the axiom to the one which is utilized in Proposition 1. However, as we saw in Proposition 2, this leads to an impossibility when there are more than two players. As we additionally saw, the IPA-flavored characterization of $KS$, too, cannot be extended to the multi-player setting.

Roughly, the intuition behind these impossibilities is as follows. The combination between an “IPA axiom” (e.g., RIC) and SINV creates a tension...
which is imposed on every pair of players. This is a type of constraint. When there is more than one pair of players, there are too many constraints.

7 Conclusion

In this paper I have presented a formal connection between the major IPA bargaining solutions and the major IPFA bargaining solutions. Based on this connection, I have derived a new axiomatization of the 2-person Nash solution. As an intermediate step, I have also derived a new axiomatization of the 2-person Kalai-Smorodinsky solution.

Bridging the gap between IPA and IPFA calls for a re-interpretation of the entire model. I have offered one such interpretation. The central idea is that each problem corresponds to possibly different set of bargainers: different problems may be thought of as involving different individuals, or involving the same individuals under different circumstances. Of course, there is limit to how far one can stretch this idea: if two sets of circumstances, or actual bargaining processes, give rise to the same “picture” in the utility space, then their solutions coincide, no matter how different they are.\footnote{Peters (1986) expresses this idea succinctly, formulating it as “Axiom 0” of the model, which states that “the solution depends only on the feasible set” (the disagreement point in Peters’ paper is normalized to the origin).}

Also, it is interesting to note the “nonmonotonicity” phenomenon with respect to the arbitrator’s outside-the-utility-space information. Under the most stringent assumption, he does not have any information of this kind, and is therefore, in particular, incapable of interpersonal utility comparisons; such an arbitrator will resort to IPFA solutions. When the arbitrator’s information is enriched, and he gets to knows the utility scales, IPA solutions emerge as
an option. Finally, if the arbitrator’s information is enriched even further, as to includes not only information about utility scales, but also the nitty-gritty details of reality, then IPFA becomes relevant again.

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8 References


