Gradual Negotiations and Proportional Solutions

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Abstract

I characterize the proportional N-person bargaining solutions by *individual rationality*, *translation invariance*, *feasible set continuity*, and a new axiom—*interim improvement*. The latter says that if the disagreement point d is known, but the feasible set is not—it may be either S or T, where $S \subset T$—then there exists a point $d' \in S$, $d' > d$, such that replacing d with d' as the disagreement point would not change the final bargaining outcome, no matter which feasible set will be realized, S or T. In words, if there is uncertainty regarding a possible expansion of the feasible set, the players can wait until it is resolved; in the meantime, they can find a Pareto improving interim outcome to commit to—a commitment that has no effect in case negotiations succeed, but promises higher disagreement payoffs to all in case negotiations fail prior to the resolution of uncertainty.

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1 Introduction

I consider Nash's bargaining problem (due to Nash (1950)) and characterize the class of its proportional solutions. A bargaining problem is defined as a pair (S, d) that satisfies the following:

- A1. $S \subset \mathbb{R}^N$, $d \in S$, and there exists an $x \in S$ such that $x > d$;[1]
- A2. S is compact and comprehensive.[2]

The set S, called the *feasible set*, consists of all the utility vectors that the players can achieve via cooperation; d_i is player i's utility in the event that cooperation fails, hence d is called the *disagreement point*. The collection of pairs (S, d) that satisfy A1-A2 is denoted by $\mathcal{B}$; a *solution* is defined to be any function $\mu\colon \mathcal{B} \to \mathbb{R}^N$ that satisfies $\mu(S, d) \in S$ for all $(S, d) \in \mathcal{B}$. Given a vector $p > \mathbf{0}$,[3] let $\mu_p(S, d) \equiv d + \epsilon p$, where ϵ is the maximal number such that the expression on the right hand side is in S. A solution μ is *proportional* if there exists a vector $p > \mathbf{0}$ such that $\mu = \mu_p$. Proportional solutions were first characterized by Kalai (1977). The *egalitarian solution*, E, corresponds to the special case $p = \mathbf{1}$; i.e., $E \equiv \mu_{\mathbf{1}}$. The reader is referred to Thomson (1994) for an excellent discussion of the bargaining model.

I characterize the proportional solutions by the following axioms, in the statements of which (S, d), (S', d'), and (T, e) are arbitrary elements of $\mathcal{B}$, and (S^n, d) is an arbitrary sequence of elements of $\mathcal{B}$ sharing the same disagreement point.

[1] Vector inequalities: xRy if and only if x_iRy_i for all i, $R \in \{>, \geq\}$; $x \gneqq y$ if and only if $x \geq y$ & $x \neq y$.

[2] The set S is *comprehensive* if for all $x, y \in S$ that satisfy $y \leq x$ it follows that $z \in S$, for every z that satisfies $y \leq z \leq x$. It is *strictly comprehensive* if in addition $P(S) \equiv \{x \in S | y \gneqq x \Rightarrow y \notin S\} = WP(S) \equiv \{x \in S | y > x \Rightarrow y \notin S\}$; that is, if its strict and weak Pareto frontiers coincide. A bargaining problem whose S is strictly comprehensive is a *strictly comprehensive bargaining problem.*

[3] $\mathbf{0} \equiv (0, \cdots, 0)$. Similarly, $\mathbf{1} \equiv (1, \cdots, 1)$.

Individual Rationality (IR): $\mu(S,d) \geq d$.

Translation Invariance (TINV): $\mu(S+t, d+t) = \mu(S,d) + t$ for all $t \in \mathbb{R}^N$.[4]

Feasible Set Continuity (S.CONT): If S^n converges to S in the Hausdorff topology and $(S,d) \in \mathcal{B}$, then $\mu(S,d) = \lim_{n\to\infty} \mu(S^n, d)$.

Interim Improvement (II): If $d = e \equiv d^*$ and $S \subset T$, then there exists a $d' > d^*$, such that $(S,d'), (T,d') \in \mathcal{B}$, $\mu(S,d') = \mu(S,d^*)$, and $\mu(T,d') = \mu(T,d^*)$.

The first two axioms are very weak; they are satisfied by all the major solutions considered in the literature. S.CONT is satisfied by all the continuous ones, and the major solutions in the literature are continuous. The fourth axiom, II, captures the following idea. Suppose that the disagreement point d is known with certainty, but the feasible set is not—it may be either S or T, where $S \subset T$. That is, it is known with certainty that all the options in S are feasible, but there are additional options, those in $T \backslash S$, the feasibility of which is uncertain. A natural course of action in this case is to "wait and see": once the uncertainty is resolved, the relevant bargaining outcome will be implemented. However, since some time passes before the realization of uncertainty, the players face a risk, as during this time negotiations may break down, an event in which they end up with the low payoffs d. A natural way to insure themselves against such an event is by signing an intermediate binding contract that specifies their payoffs in case that the bargaining procedure breaks down prior to the resolution of uncertainty. This, however, may be a difficult task, because if the interim contract affects the final outcome, then the players may behave strategically, and as a result may prefer not to sign such a contract at all. The axiom II guarantees that this is never the case: one can always find a point $d' \in S$ with $d' > d$, such that

[4] $S + t \equiv \{s + t | s \in S\}$.

replacing d with d' as the disagreement point would not change the final bargaining outcome, independent of the realization of uncertainty.[5]

The rest of the paper is organized as follows: Section 2 contains the main result and Section 3 offers a discussion that consists of a sequence of subsections, each focusing on a different aspect of that result.

2 The main result

Consider the following axioms, in the statements of which (S, d) and (T, e) are arbitrary elements of $\mathcal{B}$, and (S, d_n) is an arbitrary sequence of elements of $\mathcal{B}$ sharing the same feasible set.

Weak Pareto Optimality (WPO): $\mu(S, d) \in WP(S)$.[6]

Weak Disagreement Point Continuity (W.D.CONT): If $\mu(S, d_n) = x$ for all n, $d_n \to d$ and $(S, d) \in \mathcal{B}$, then $\mu(S, d) = x$.

Strong Individual Rationality (S.IR): $\mu(S, d) > d$.

[5] Axioms concerning uncertainty regarding components of the bargaining problem have been studied extensively in the literature. The two most notable examples are Perles and Maschler's *super additivity* (see Perles and Maschler (1981)) and Chun and Thomson's *disagreement point concavity* (see Chun and Thomson (1990)). Holding the disagreement point d fixed, the former requires a mixture of two feasible sets, S and T, to lead to a solution point that lies above the respective mixture of the solutions of (S, d) and (T, d); the latter imposes the analogous requirement on mixtures of two disagreement points, when the feasible set is fixed. It is worth noting that as opposed to these axioms, II does not assume commonly known probabilities. That is, it accommodates the case where player i believes that the options in $T \backslash S$ will become available with probability p_i, where p_i is an idiosyncratic (and not necessarily commonly known) value.

[6] A natural strengthening of this axiom is Pareto Optimality (PO), which requires $\mu(S, d) \in P(S)$ for all $(S, d) \in \mathcal{B}$.

Independence of Non Individually Rational Alternatives (INIR): $\mu(S, d) = \mu(S_d, d)$, where $S_d \equiv \{x \in S | x \geq d\}$.

Monotonicity (MON): If $d = e \equiv d^*$ and $S \subset T$, then $\mu(S, d^*) \leq \mu(T, d^*)$.

Step by Step Negotiations (SSN): If $d = e \equiv d^*$, $S \subset T$, and $(T, \mu(S, d^*)) \in \mathcal{B}$, then $\mu(T, d^*) = \mu(T, \mu(S, d^*))$.

The first axiom in the list is obvious; the second axiom is a weaker version of the more common *disagreement point continuity*, which makes the same requirement, but without the restriction to sequences of elements that give rise to the same solution point; the third and fourth axioms, which are due to Roth (1977) and Peters (1986) respectively,[7] strengthen IR; the fifth axiom, which is due to Kalai (1977), says that if more options become available, no one should get hurt; the sixth axiom, also due to Kalai (1977), says that whenever the players face two nested problems with a common disagreement point, they can first solve the smaller (and presumably simpler) problem, and then regard its solution as the disagreement point of the "continuation problem."

Lemma 1. *S.CONT and TINV imply W.D.CONT.*

Proof. Let μ be a solution that satisfies S.CONT and TINV. Suppose that $(S, d_n) \in \mathcal{B}$ and $\mu(S, d_n) = x$ for all n, $d_n \to d$ and $(S, d) \in \mathcal{B}$. By TINV, $\mu(S - d_n, \mathbf{0}) = x - d_n$. By S.CONT, $\mu(S - d, \mathbf{0}) = x - d$. Applying TINV again gives $\mu(S, d) = x$. □

Let $\mathbb{M}$ denote the set of solutions that satisfy II, IR, TINV, and S.CONT.

Lemma 2. *Every $\mu \in \mathbb{M}$ satisfies SSN.*

[7] The aforementioned paper by Peters was published as a book chapter in 2010, and hence appears with the 2010 date on the reference list.

Proof. Let $\mu \in \mathbb{M}$. By S.CONT, it suffices to prove that SSN holds on the domain of strictly comprehensive problems. Let $(S, d), (T, d') \in \mathcal{B}$ be such that $d' = d$, $S \subset T$ and are both strictly comprehensive, and $(T, \mu(S, d)) \in \mathcal{B}$. By Lemma 1, μ satisfies W.D.CONT. By II and W.D.CONT, it follows that there exists a sequence $\{d_n\} \subset S_d$, such that $\mu(S, d_n) = \mu(S, d)$ and $\mu(T, d_n) = \mu(T, d)$ for all n; let $d^* \equiv \lim_{n \to \infty} d_n$ (existence of the limit follows from the compactness of S_d). I argue that $d^* \in WP(S)$. To see this, denote by L the set of all limits of these sequences. By the aforementioned existence argument, $L \neq \emptyset$. Define the binary relation $\succ$ on L by: $l \succ l'$ if and only if $l_1 > l'_1$. Since $L \subset S_d$ and the latter is compact, there exists a $\succ$-maximal element in the closure of L, l^*, which belongs to S_d. I argue that $l^* \in WP(S)$. Otherwise $(S, l^*) \in \mathcal{B}$, and therefore, by II, there exists an $l' > l^*$ such that $(S, l') \in \mathcal{B}$, $\mu(S, l') = \mu(S, l^*)$, and $\mu(T, l') = \mu(T, l^*)$, in contradicts to the $\succ$-maximality of l^*. Now, since S is strictly comprehensive, $d^* \in P(S)$. By IR, $d_n \leq \mu(S, d_n) = \mu(S, d)$, hence $d^* \leq \mu(S, d)$. Since $d^* \in P(S)$, $d^* = \mu(S, d)$. Since $(T, \mu(S, d)) = (T, d^*) \in \mathcal{B}$, and $\mu(T, d_n) = \mu(T, d)$ for all n, W.D.CONT implies that $\mu(T, d^*) = \mu(T, d)$. Therefore, $(T, \mu(S, d)) = (T, d)$. □

Lemma 3. *Every $\mu \in \mathbb{M}$ satisfies WPO.*

Proof. Let $\mu \in \mathbb{M}$ and let $(S, d) \in \mathcal{B}$. By the argument from Lemma 2 there exists an increasing sequence $\{d_n\}$ such that $\mu(S, d_n) = \mu(S, d)$ for all n, that converges to a limit $d^* \in WP(S)$. By IR, $d_n \leq \mu(S, d_n) = \mu(S, d)$, therefore $d^* \leq \mu(S, d)$. Therefore, $\mu(S, d) \in WP(S)$. □

Lemma 4. *Every $\mu \in \mathbb{M}$ satisfies INIR.*

Proof. Let $\mu \in \mathbb{M}$ and let $(S, d) \in \mathcal{B}$. By S.CONT we may assume that S is strictly comprehensive. Let $x \equiv \mu(S, d)$ and $\mu(S_d, d) \equiv y$. By the arguments from Lemma 2, there exists an increasing sequence $\{d_n\} \subset S_d$, with a limit $d^* \in P(S_d)$. By the arguments from Lemma 2, both $d^* = x$ and $d^* = y$, hence $x = y$. □

Given $r > 0$, let $\Delta_r \equiv \{x \in \mathbb{R}^N_+ | \sum_{i=1}^N x_i \leq r\}$.

Lemma 5. *Let* $\mu \in \mathbb{M}$. *Then* $\mu(\Delta_{kr}, \mathbf{0}) = k\mu(\Delta_r, \mathbf{0})$ *for all* $r > 0$ *and* $k \in \mathbb{N}$.

Proof. Make the assumptions of the lemma and let $r > 0$. The statement of the lemma is trivial for $k = 1$; I will prove that it holds for $k \geq 2$ by induction.

$k = 2$. Here, we need to prove $\mu(\Delta_{2r}, \mathbf{0}) = 2\mu(\Delta_r, \mathbf{0})$. By SSN and TINV, the left hand side equals $\mu(\Delta_{2r}, \mu(\Delta_r, \mathbf{0})) = \mu(\Delta_{2r} - \mu(\Delta_r, \mathbf{0}), \mathbf{0}) + \mu(\Delta_r, \mathbf{0})$; therefore, we need to prove $\mu(\Delta_{2r} - \mu(\Delta_r, \mathbf{0}), \mathbf{0}) = \mu(\Delta_r, \mathbf{0})$. By INIR, then, it is enough to prove that $\mu([\Delta_{2r} - \mu(\Delta_r, \mathbf{0})] \cap \mathbb{R}^N_+, \mathbf{0}) = \mu(\Delta_r \cap \mathbb{R}^N_+, \mathbf{0})$. This is indeed the case, because by WPO $[\Delta_{2r} - \mu(\Delta_r, \mathbf{0})] \cap \mathbb{R}^N_+ = \Delta_r \cap \mathbb{R}^N_+$.[8]

The inductive step: Suppose that the lemma is true for $(k-1)$, where $k \geq 2$. By SSN and TINV, $\mu(\Delta_{kr}, \mathbf{0}) = \mu(\Delta_{kr} - \mu(\Delta_{(k-1)r}, \mathbf{0}), \mathbf{0}) + \mu(\Delta_{(k-1)r}, \mathbf{0})$. By the induction's hypothesis $\mu(\Delta_{(k-1)r}, \mathbf{0}) = (k-1)\mu(\Delta_r, \mathbf{0})$. Therefore, we need to prove that $\mu(\Delta_{kr} - \mu(\Delta_{(k-1)r}, \mathbf{0}), \mathbf{0}) = \mu(\Delta_r, \mathbf{0})$. By INIR, therefore, it is enough to prove that $\mu([\Delta_{kr} - \mu(\Delta_{(k-1)r}, \mathbf{0})] \cap \mathbb{R}^N_+, \mathbf{0}) = \mu(\Delta_r \cap \mathbb{R}^N_+, \mathbf{0})$. This is indeed the case, because by WPO $[\Delta_{kr} - \mu(\Delta_{(k-1)r}, \mathbf{0})] \cap \mathbb{R}^N_+ = \Delta_r \cap \mathbb{R}^N_+$. □

Lemma 6. *Every* $\mu \in \mathbb{M}$ *satisfies MON.*

Proof. Let $\mu \in \mathbb{M}$. By S.CONT, it suffices to prove that MON holds for strictly comprehensive problems. Let then $(S, d), (T, d) \in \mathcal{B}$ be two strictly comprehensive problems such that $S \subset T$. Assume by contradiction that there is some i such that $x_i > y_i$, where $x \equiv \mu(S, d)$ and $y \equiv \mu(T, d)$. By the arguments from Lemma 2, there exists a sequence $\{d_n\} \subset S$ such that $\mu(S, d_n) = \mu(S, d)$ and $\mu(T, d_n) = \mu(T, d)$ for all n, and $d^* \equiv \lim_{n\to\infty} d_n \in WP(S) = P(S)$. By IR, $d_n \leq x$ for all x, hence $d^* \leq x$; therefore, $d^* = x$. By Lemma 1, μ satisfies W.D.CONT, and therefore $y = \mu(T, d) = \mu(T, d^*) = \mu(T, x)$. However, $x_i > y_i$ contradicts IR. □

Let $p_\mu(r) \equiv \mu(\Delta_r, \mathbf{0})$.

[8] To see this set-equality, consider first $x \in [\Delta_{2r} - \mu(\Delta_r, \mathbf{0})] \cap \mathbb{R}^N_+$. We have that $x + \mu(\Delta_r, \mathbf{0}) \in \Delta_{2r}$; since, by WPO, $\sum_i \mu_i(\Delta_r, \mathbf{0}) = r$ it follows that $\sum_i x \leq r$, which, together with the fact that $x \geq \mathbf{0} \cap \mathbb{R}^N_+$ implies that $x \in \Delta_r$. Conversely, for $x \in \Delta_r \cap \mathbb{R}^N_+$ we have $\sum_i x_i \leq r$ and therefore $\sum_i x_i + \sum_i \mu_i(\Delta_r, \mathbf{0}) \leq 2r$, so $x \in [\Delta_{2r} - \mu(\Delta_r, \mathbf{0})] \cap \mathbb{R}^N_+$.

Lemma 7. *Let $\mu \in \mathbb{M}$. Then $p_\mu(.)$ is homogeneous of degree one. That is, $p_\mu(r) = rp_\mu(1)$ for every $r > 0$.*

Proof. Let $\mu \in \mathbb{M}$ and $r > 0$. There exists a positive integer N such that $r > (\frac{1}{2})^n$ for all $n \geq N$. For every integer $n \geq N$ there exists a unique integer $k = k(n)$ such that $k(\frac{1}{2})^n \leq r \leq (k+1)(\frac{1}{2})^n$. If there is an n such that one of these inequalities is satisfied as equality, then, by Lemma 5, we are done.[9] Suppose, on the other hand, that $k(\frac{1}{2})^n < r < (k+1)(\frac{1}{2})^n$ for every $n \geq N$. Let $S_n \equiv \Delta_{k(\frac{1}{2})^n}$ and $S_n^+ \equiv \Delta_{(k+1)(\frac{1}{2})^n}$. There exists an N such that $r > (\frac{1}{2})^n$ for all $n \geq N$. For every integer $n \geq N$ there exists a unique integer $k = k(n)$ such that $k(\frac{1}{2})^n \leq r \leq (k+1)(\frac{1}{2})^n$. If there is an n such that one of these inequalities is satisfied as equality, then, by Lemma 5, we are done.[10] Suppose, on the other hand, that $k(\frac{1}{2})^n < r < (k+1)(\frac{1}{2})^n$ for every $n \geq N$. Let $S_n \equiv \Delta_{k(\frac{1}{2})^n}$ and $S_n^+ \equiv \Delta_{(k+1)(\frac{1}{2})^n}$. By MON,

$$\mu(S_n, \mathbf{0}) \leq p_\mu(r) \leq \mu(S_n^+, \mathbf{0}). \tag{1}$$

Note that $\mu(S_n, \mathbf{0}) = p_\mu(k(\frac{1}{2})^n) = kp_\mu((\frac{1}{2})^n) = k\mu(\Delta_{(\frac{1}{2})^n}, \mathbf{0})$; the first and third equalities follow from the definition of p_μ, and the second also involves Lemma 5. Also, $p_\mu(1) = \mu(\Delta_1, \mathbf{0}) = \mu(\Delta_{2^n(\frac{1}{2})^n}, \mathbf{0}) = 2^n\mu(\Delta_{(\frac{1}{2})^n}, \mathbf{0})$ follows from Lemma 5. Therefore, $\mu(\Delta_{(\frac{1}{2})^n}, \mathbf{0}) = (\frac{1}{2})^n p_\mu(1)$ for every $n \geq N$. Therefore, $\mu(S_n, \mathbf{0}) = k(\frac{1}{2})^n p_\mu(1)$. Similarly, $\mu(S_n^+, \mathbf{0}) = (k+1)(\frac{1}{2})^n p_\mu(1)$. Plugging this into (1) gives $k(\frac{1}{2})^n p_\mu(1) \leq p_\mu(r) \leq (k+1)(\frac{1}{2})^n p_\mu(1)$. Taking $n \to \infty$ gives $rp_\mu(1) \leq p_\mu(r) \leq rp_\mu(1)$, so $p_\mu(r) = rp_\mu(1)$. □

[9] To see this, suppose that $k(\frac{1}{2})^n = r$. Then, on the one hand, $p_\mu(r) = p_\mu(k(\frac{1}{2})^n) = \mu(\Delta_{k(\frac{1}{2})^n}, \mathbf{0}) = k\mu(\Delta_{(\frac{1}{2})^n}, \mathbf{0})$, where the last equality is by Lemma 5. On the other hand, $rp_\mu(1) = k(\frac{1}{2})^n\mu(\Delta_1, \mathbf{0}) = k(\frac{1}{2})^n\mu(\Delta_{2^n(\frac{1}{2})^n}, \mathbf{0}) = k\mu(\Delta_{(\frac{1}{2})^n}, \mathbf{0})$, where the last equality is by Lemma 5.

[10] To see this, suppose that $k(\frac{1}{2})^n = r$. Then, on the one hand, $p_\mu(r) = p_\mu(k(\frac{1}{2})^n) = \mu(\Delta_{k(\frac{1}{2})^n}, \mathbf{0}) = k\mu(\Delta_{(\frac{1}{2})^n}, \mathbf{0})$, where the last equality is by Lemma 5. On the other hand, $rp_\mu(1) = k(\frac{1}{2})^n\mu(\Delta_1, \mathbf{0}) = k(\frac{1}{2})^n\mu(\Delta_{2^n(\frac{1}{2})^n}, \mathbf{0}) = k\mu(\Delta_{(\frac{1}{2})^n}, \mathbf{0})$, where the last equality is by Lemma 5.

Given $r > 0$, let $C(r) \equiv \{x \in \mathbb{R}^N_+ | x \leq p(r)\}$.[11]

Lemma 8. *Let $\mu \in \mathbb{M}$ and let $r > 0$. Then $\mu(C(r), \boldsymbol{0}) = p_\mu(r)$.*

Proof. Make the assumptions of the lemma. Assume by contradiction that $x \neq p_\mu(r)$, where $x \equiv \mu(C(r), \mathbf{0})$. By SSN, $p_\mu(r) = \mu(\Delta_r, x)$. By II, there exists a $y > x$ such that $(\Delta_r, y) \in \mathcal{B}$ and $\mu(\Delta_r, y) = p_\mu(r)$. Therefore, by IR, $p_\mu(r) \geq y > x$, in contradiction to WPO. □

Armed with the lemmas, we can turn to the main result.

Theorem 1. *A solution belongs to $\mathbb{M}$ if and only if it is proportional.*

Proof. It is clear that every proportional solution is in $\mathbb{M}$. Conversely, let $\mu \in \mathbb{M}$. I will prove that $\mu = \mu_p$, where $p \equiv p_\mu(1)$; the combination of IR and II implies S.IR, hence $p > \mathbf{0}$.

By S.CONT and TINV, it is enough to prove that μ coincides with μ_p on the the class of strictly comprehensive bargaining problems with disagreement point $\mathbf{0}$. Let then $(S, \mathbf{0})$ be such a problem. Let $r > 0$ be the unique number such that $rp_\mu(1) = \mu_p(S, \mathbf{0})$. Assume by contradiction that $\mu(S, \mathbf{0}) \neq \mu_p(S, \mathbf{0})$. Since $\mu_p(S, \mathbf{0}) \in WP(S) = P(S)$, there exists an i such that $\mu_i(S, \mathbf{0}) < \mu_{p,i}(S, \mathbf{0}) = rp_{\mu,i}(1)$. Let $r' < r$ be such that

$$\mu_i(S, \mathbf{0}) < r' p_{\mu,i}(1). \tag{2}$$

We have that $\mu(S, \mathbf{0}) = \mu(S, \mu(C(r'), \mathbf{0})) \geq \mu(C(r'), \mathbf{0}) = p_\mu(r') = r' p_\mu(1)$; the first equality is by SSN, the inequality is by IR, and the following equalities are by Lemma 8 and Lemma 7, respectively. This, of course, contradicts (2). □

[11] A more complete notation would be "$C_\mu(r)$" instead of "$C(r)$." The extra subscript is skipped in order to make the notation a bit lighter.

3 Discussion

3.1 Independence of the axioms

Below are four solutions, each of which violates exactly one of the axioms from Theorem 1 and satisfies the remaining three.

1. All but IR: Consider $-E(S,d) \equiv d - x \cdot \mathbf{1}$, where x is the maximal number such that the right hand side is in S.

2. All but II: The Kalai-Smorodinsky solution (due to Kalai and Smorodinsky (1975)). This solution assigns to each $(S,d) \in \mathcal{B}$ the point $(1-\theta)d + \theta a(S,d)$, where θ is the maximal number such that the aforementioned expression is in S, where $a_i(S,d) \equiv \max\{x_i | x \in S_d\}$.

3. All but TINV: Let $N = 2$.[12] Fix an $\varepsilon > 0$. Let $D \equiv \{x \in \mathbb{R}^2 | x_1 = x_2\}$. Denote by B the open band of width ε around the plane's diagonal, D; i.e., $B \equiv \{x \in \mathbb{R}^2 : ||x - D|| < \varepsilon\}$. Let $\hat{\mu}$ be the following solution,

$$\hat{\mu}(S,d) \equiv \begin{cases} E(S,d) & \text{if } d \in B \\ \mu_{(2,1)}(S,d) & \text{if } d \notin B \text{ and } d_1 > d_2 \\ \mu_{(1,2)}(S,d) & \text{if } d \notin B \text{ and } d_1 < d_2 \end{cases}$$

It is easy to see that $\hat{\mu}$ satisfies all the axioms of Theorem 1 besides TINV. This solution captures the idea that the bargaining outcome may be sensitive to the outside options: if the outside options are sufficiently close, then the solution splits the surplus equally; otherwise, the stronger player is being favored in terms of receiving a larger proportion of the surplus.

[12] The generalization of this example to an arbitrary N is easy.

4. All but S.CONT. Let $N = 2$ and consider the *lexicographic extension of* E, E_{lex}, which is defined as follows.[13] For every $(S, d) \in \mathcal{B}$ there exists at most one player i such that the i-th coordinate of $E(S, d)$ can be increased without decreasing coordinate $j \neq i$ and without leaving S; E_{lex} is obtained by applying the maximal payoff increase to this i if such an i exists, and $E_{lex} = E$ otherwise. It is easy to see that this solution satisfies all the axioms besides S.CONT.

Example 4 may lead one to suspect that once S.CONT is deleted from the list of axioms from Theorem 1, a characterization of the lexicographic extensions of the proportional solutions is obtained. This is not true; in fact, this is not true even if PO is added to the axiom list. I conclude this subsection with an example of a 2-person solution, which is not proportional, and that satisfies PO, IR, TINV, and II.

Let $S_0 = \{x \in \mathbb{R}^2_+ | x_1^2 + x_2^2 \leq 1\}$ and consider the following solution,

$$\tilde{\mu}(S, d) \equiv \begin{cases} (\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}) + t & \text{if there is a } t \text{ such that } S = S_0 + t \text{ and } d < (\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}) + t \\ E_{lex}(S, d) & \text{otherwise} \end{cases}$$

3.2 Other domains

The feasible sets are allowed to be non-convex. The are many reasons why non-convexities may arise in bargaining. For example, the players may not have access to a randomization device, and even if they do they may prefer not to use it, because they may view some issues as too important to be decided on by gambling. Moreover, it may be that the utilities in the bargaining problem are not v-N.M utilities. All the analysis above remains equally valid if convexity is added (to A1-A2 above). In other words, none of the arguments above invokes non-convex feasible sets.[14]

[13] The generalization to $N > 2$ is straightforward. This solution has been studied by Chun (1989). The lexicographic extension of any other proportional solution μ_p is defined analogously.

[14] For further works on bargaining with non-convex feasible sets, see Anant et. al. (1990), Conley and Wilkie (1991), Herrero (1989), Hougaard and Tvede (2003), Hougaard and Tvede (2010), Peters

Similarly, the analysis remains valid if compactness is replaced by unboundedness from below,[15] coupled with compactness of S_d. Namely, if *free disposal of utility* is allowed. The only thing that needs to be changed is Example 1, since the solution it describes is not well-defined when the feasible set is unbounded from below. The modification $-\tilde{E}(S,d) \equiv E(S,d) - \mathbf{1}$ does the job.

3.3 II vs. SSN

II and SSN share a similar flavor: both describe a "graduality" property. They are, however, logically incomparable.

Proposition 1. *SSN and II are logically incomparable.*

Proof. Consider the *disagreement solution* $\mu(S,d) \equiv d$. It is immediate that it satisfies SSN and violates II. The solution E_{lex} satisfies II but not SSN. □

As the following proposition shows, the fact that S.IR is violated in the example of a solution that satisfies SSN but violates II is not coincidental.

Proposition 2. *SSN and S.IR imply II.*

Proof. Let μ satisfy S.IR and SSN and let (S,d) and (T,d) be two elements of $\mathcal{B}$ with $S \subset T$, who share the disagreement point d. We need to find a $d' > d$ such that $(S,d') \in \mathcal{B}$ and $\mu(X,d') = \mu(X,d)$ for both $X \in \{S,T\}$. Clearly we can pick a subset $Q \subset S$ such that $(Q,d) \in \mathcal{B}$ and $WP(Q) \cap WP(S) = \emptyset$. Let $d' \equiv \mu(Q,d)$. By S.IR, $d' > d$. By SSN, $\mu(X,d') = \mu(X,d)$ for both $X \in \{S,T\}$. □

Proposition 2 can be viewed as saying that even though SSN and II are incomparable, the former is "almost stronger" than the latter.

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and Vermeulen (2010), Serrano and Shimomura (1998), Xu and Yoshihara (2006), and Zho (1996).

[15] i.e., $x \in S$ and $y \leq x$ implies $y \in S$.

3.4 The utility space vs. the physical reality

The motivating story for II is that it refers to uncertainty, the resolution of which reveals the feasible set, S or T. However, since the axiom (and, in fact, the entire bargaining model) is formulated in the utility space—the aforementioned uncertainty is only at the level of interpretation—the following critique presents itself: the axiom can be applied to any pair of problems with a common disagreement point, even ones that have nothing to do with one another in terms of the underlying bargaining process. This is certainly a legitimate criticism, but it applies equally to any axiom that refers to the underlying bargaining process (SSN, for example); II, in this respect, is not special.

3.5 Contingent contracts

Why do the players need to sign a contract that updates the disagreement point? If contracts are allowed, why not sign a contingent contract (i.e., a pair of contracts) that specifies one agreement for the event that S is realized and another agreement for the event that T is realized? The response to this question is threefold. First, nothing in II excludes the possibility of such contingent contracts; II, however, says that they are not necessary for the characterization of the proportional solutions. Secondly, in terms of practicality, contingent contracts may be complicated and costly, while agreements about the default outcome may be less complicated. Thirdly, contingent contracts are executed *after* the resolution of uncertainty—they do not offer the ex ante protection regarding what happens if negotiations break down prior to that resolution, which is precisely the protection offered by II.

3.6 Early agreements

Related to the previous subsection is the fact that II and contingent contracts alike involve taking action prior to the resolution of some uncertainty. Such early action has a solid place in the existing literature; in particular, there is extensive reference in the literature to *early agreements*. Both Myerson (1981) and Perles and Maschler (1981) express its desirability axiomatically: the former's *concavity* and the latters' *super additivity* demand that "mixing" two feasible sets should result in a feasible set whose solution-point dominates the mixture of the solution-points of the mixed sets. Consequently, no one should be hurt by early agreement.

II is of a different nature. Whereas the aforementioned axioms express the benefits of early agreements, in II there are no benefits: the improvement, measured by the difference $d^* - d'$, is only realized if negotiations break down—an event that does not occur. Additionally, as opposed to the aforementioned axioms, II does not assume expected utility (see subsection 3.2 above).

3.7 The strength of II

II is a strong axiom. One may criticize it on the basis that it is *too* strong. I believe that such criticism would be unjustified. First, the "too strong" claim can be made towards many of the axioms in the mainstream bargaining literature: Myerson's concavity, Perles and Maschler's super additivity, and Kalai's monotonicity are all examples in place. Secondly, II's strength can be viewed as a blessing, not as a curse: one may draw the conclusion that the property which is described by this axiom is *the* defining feature of the proportional solutions. This conclusion, I believe, is especially appealing when viewed in light of the extreme weakness of the other axioms which are involved in Theorem 1.

3.8 Relaxing II

In light of the strength of II, one may seek ways to weaken it. One way to do so is to restrict the axiom to situations where $S = T$; that is, to situations where there is no uncertainty regarding the feasible set, and the whole bite of the axiom is that it guarantees the existence of a better disagreement point that does not change the final outcome. The resulting axiom is extremely weak; in particular, it is satisfied by all the major solutions from the literature. Additionally, such a weak version is less interesting at the level of interpretation: if there is no need to wait for the resolution of uncertainty, then the risk of negotiation breakdown—which is the motivation for signing the interim contract from II—does not even present itself.

An alternative weakening of II is to replace its strict inequality by a weak one. Namely, to demand from the interim disagreement point to satisfy only $d' \gneqq d^*$, not $d' > d^*$. Call the resulting axiom *weak interim improvement* (W.II). Replacing II by W.II in Theorem 1 results in a characterization of the solutions of the form μ_p for $p \gneqq \mathbf{0}$. In particular, this class of solutions includes the *dictatorial solutions*. Formally, the *i-th dictatorial solution*, D_i, is given by $D_i(S,d) \equiv (a_i(S,d), d_{-i})$. The exclusion of dictatorship when W.II is strengthened to II is not a mere technicality, but, as we will see in the subsection below, has a significant economic content.

3.9 Interpersonal utility comparisons

The central philosophical idea that the proportional solutions express is that of *interpersonal utility comparisons*. The alternative idea, of *interpersonal incomparability*, when stated axiomatically, goes as follows.

Independence of Equivalent Utility Representations (IEUR): If $\lambda\colon \mathbb{R}^N \to \mathbb{R}^N$ is a positive affine transformation, then $\mu(\lambda \circ S, \lambda \circ d) = \lambda \circ \mu(S,d)$.[16,17]

That II captures the essence of the proportional solutions (see subsection 3.7 above) receives further support, in form of the following result.

Proposition 3. *There does not exist a solution that satisfies II, INIR, and IEUR.*

[16] $\lambda = (\lambda_1, \cdots, \lambda_N)$ is a positive affine transformation if the following holds for each i: $\lambda_i(x) = \alpha_i x + \beta_i$, where $\alpha_i > 0$.

[17] TINV is obtained as a special case of this axiom, where $\alpha_1 = \alpha_2 = \cdots = \alpha_N = 1$.

Proof. For simplicity, I prove the result for $N = 2$. Assume by contradiction that there exists a solution that satisfies the three axioms. Let $R = \{x \in \mathbb{R}^2_+ | x \leq (1,1)\}$ and let $R' = \{x \in \mathbb{R}^2_+ | x \leq (2,1)\}$. By II, there exists a point $(a,b) > \mathbf{0}$, such that replacing $\mathbf{0}$ with (a,b) will not change the outcome in either of the problems $(R, \mathbf{0})$ and $(R', \mathbf{0})$. Applying IEUR and INIR to the problems $(R, \mathbf{0})$ and $(R, (a,b))$, we see that (a,b) must satisfy $\frac{1-b}{1-a} = 1$. Applying the same argument to $(R', \mathbf{0})$ and $(R', (a,b))$, we conclude that $\frac{1-b}{1-a} = 2$, a contradiction. □

This result emphasizes that the difference between II and W.II is indeed not trivial. The latter, as opposed to the former, is consistent with the combination of IEUR and INIR. For example, the dictatorial solutions satisfy W.II, IEUR, and INIR. This may lead one to suspect that the combination of these three axioms results in a characterization of the dictatorial solutions. As the following 2-person example shows, this is not the case.

$$\mu^*(S,d) \equiv \begin{cases} x & \text{if } x \text{ is the unique kink in the relative interior of } WP(S_d) \\ D_1(S,d) & \text{otherwise} \end{cases}$$

It is easy to see that μ^* satisfies W.II, IEUR, and INIR.

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4 References

Anant, T., Basu, K., and Mukherji, B. (1990), Bargaining without convexity, *Economics Letters*, **33**, 115-119.

Chun, Y. (1989), Lexicographic egalitarian solution and uncertainty in the disagreement point, *Mathematical Methods of Operations Research*, **33**, 259-266.

Chun, Y. and Thomson, W. (1990), Bargaining with uncertain disagreement point, *Econometrica*, **58**, 951-959.

Conley, J., and Wilkie, S. (1991), The bargaining problem without convexity, *Economics Letters*, **36**, 365-369.

Herrero, M.J. (1989), The Nash program: nonconvex bargaining problems, *Journal of Economic Theory*, **49**, 266-277.

Hougaard, J.L., and Tvede, M. (2003), Nonconvex n-person bargaining: efficient maxmin solutions, *Economic Theory*, **21**, 81-95.

Hougaard, J.L., and Tvede, M. (2010), n-person non-convex bargaining: efficient proportional solutions, *Operations Research Letters*, **38**, 536-538.

Kalai, E. (1977), Proportional solutions to bargaining situations: interpersonal utility comparisons, *Econometrica* **45**, 1623-1630.

Kalai, E. and Smorodinsky, M. (1975), Other solutions to Nash's bargaining problem, *Econometrica* **43**, 513-518.

Myerson, R.B. (1981), Utilitarianism, egalitarianism, and the timing effect in social choice problems, *Econometrica*, **49**, 883-897.

Nash, J. F. (1950), The bargaining problem, *Econometrica* **18** , 155-162.

Perles, M. A. and Maschler, M. (1981), The super-additive solution for the Nash bargaining problem, *International Journal of Game Theory*, **10**, 163-193.

Peters, H. (2010), Characterizations of bargaining solutions by properties of their status quo sets, in Collective Decision Making: Views from Social Choice and Game Theory (A.Van Deemen and A. Rusinowska, eds.), Theory and Decision Library Series C, Springer, Berlin Heidelberg, 231-247.

Peters, H., and Vermeulen, D. (2010), WPO, COV and IIA bargaining solutions for non-convex bargaining problems, *International Journal of Game Theory*, forthcoming.

Roth, A. (1977), Individual rationality and Nash's solution to the bargaining problem, *Mathematics of Operations Research*, **2**, 64-65.

Serrano, R., and Shimomura, K-I. (1998), Beyond Nash bargaining theory: the Nash set, *Journal of Economic Theory*, **83**, 286-307.

Thomson, W. (1994), Cooperative models of bargaining, in R. J. Aumann and S. Hart (eds.) Handbook of game theory with economic applications, vol.2, Elsevier, New York.

Xu, Y., and Yoshihara, N. (2006), Alternative characterizations for three bargaining solutions for nonconvex problems, *Games and Economic Behavior*, **57**, 86-92.

Zhou, L. (1996), Nash bargaining theory with nonconvex problems, *Econometrica*, **65**, 681-685.