Disagreement in bargaining and the Nash solution

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Abstract

Roth (1979) showed that when individual rationality (IR) replaces weak Pareto optimality (WPO) in the axiom list of Nash’s (1950) characterization of the Nash bargaining solution, exactly two solutions become admissible: the Nash solution and the disagreement solution. Here I show that instead of IR, WPO can be replaced by disagreement domination (DID), and the same conclusion would be reached. DID—an axiom which is new to the literature—requires solution payoffs never to be Pareto dominated by the disagreement point. DID is logically weaker than IR and there exist meaningful bargaining solutions, such as the utilitarian solution, that violate IR but satisfy DID.

Keywords: Axioms; Bargaining; Disagreement; Nash solution.

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1 Introduction

A bargaining problem is a pair consisting of a set of payoffs—the feasible set—and a specific point in that set, the disagreement point. The interpretation is the following:

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two players need to choose one point (a utility allocation) from the feasible set; if they agree on \( x \) then the utility payoff for player \( i \) is \( x_i \), while failing to reach an agreement leads to the implementation the disagreement point payoffs. A bargaining solution is a selection—a rule that picks a unique payoff vector from the feasible set of every problem.

Nash (1950) characterized his solution—the maximizer of the “Nash product”—on the basis of four axioms: \textit{weak Pareto optimality}, \textit{independence of equivalent utility representations}, \textit{symmetry}, and \textit{independence of irrelevant alternatives}. Roth (1979) showed that when \textit{individual rationality} replaces weak Pareto in the aforementioned list, exactly two solutions become admissible: the Nash solution and the disagreement solution—the solution that picks, for every problem, its disagreement point. A consequence of this fact is that in the presence of the other axioms, Nash’s solution is pinned down by an extremely mild requirement—“not disagreement.”

Here I present a strengthening of Roth’s theorem. I show that weak Pareto can be replaced by \textit{disagreement domination}. This axiom—which is new to the literature—requires solution payoffs never to be Pareto dominated by the disagreement point.

2 Model and axioms

A \textit{bargaining problem} is defined as a pair \((S, d)\), where \( S \subset \mathbb{R}^2 \) is the \textit{feasible set}, representing all possible (v-N.M) utility agreements between the two players, and \( d \in S \), the \textit{disagreement point}, is a point that specifies their utilities in case they do not reach a unanimous agreement on some point of \( S \). It is assumed that \( S \) is compact and convex, and that there is some \( x \in S \) with \( x > d \).\(^1\) Denote by \( \mathcal{B} \) the collection of all such pairs \((S, d)\). A \textit{solution} is any function \( \mu: \mathcal{B} \to \mathbb{R}^2 \) that satisfies \( \mu(S, d) \in S \) for all \((S, d) \in \mathcal{B} \). Given a feasible set \( S \), the \textit{weak Pareto frontier} of \( S \) is

\(^1\)Vector inequalities: \( xRy \) if and only if \( x_iRy_i \) for both \( i \in \{1, 2\} \), \( R \in \{>, \geq\} \); \( x \geq y \) if and only if \( x \geq y \) & \( x \neq y \).
$WP(S) \equiv \{ x \in S : y > x \Rightarrow y \notin S \}$. The set of individually rational points in $(S, d)$ is $S_d \equiv \{ x \in S : x \geq d \}$.

The *Nash solution*, $N$, due to Nash (1950), is defined to be the unique maximizer of $(x_1 - d_1) \times (x_2 - d_2)$ over $S_d$. Nash (1950) showed that $N$ is the unique solution that satisfies the following four axioms, in the statements of which $(S, d)$ and $(T, e)$ are arbitrary problems.

**Weak Pareto Optimality** (WPO): $\mu(S, d) \in WP(S)$.

Let $F_A$ denote the set of positive affine transformations from $\mathbb{R}$ to itself.\(^2\)

**Independence of Equivalent Utility Representations** (IEUR): $f = (f_1, f_2) \in F_A \times F_A \Rightarrow f \circ \mu(S, d) = \mu(f \circ S, f \circ d)$.\(^3\)

Let $\pi(a, b) \equiv (b, a)$.

**Symmetry** (SY): $[\pi \circ S = S] \& [\pi \circ d = d] \Rightarrow \mu_1(S, d) = \mu_2(S, d)$.

**Independence of Irrelevant Alternatives** (IIA): $[S \subset T] \& [d = e] \& [\mu(T, e) \in S] \Rightarrow \mu(S, d) = \mu(T, e)$.

Consider the following axiom, in the statement of which $(S, d)$ is an arbitrary problem.

**Individual Rationality** (IR): $\mu(S, d) \geq d$.

A solution that satisfies IR is simply referred to as *individually rational*. Roth (1979)\(^2\) i.e., the set of functions $f$ of the form $f(x) = \alpha x + \beta$, where $\alpha > 0$.

\(^3\)If $f_i : \mathbb{R} \to \mathbb{R}$ for each $i = 1, 2$, $x \in \mathbb{R}^2$, and $A \subset \mathbb{R}^2$, then: $(f_1 \circ f_2) \circ x \equiv (f_1(x_1), f_2(x_2))$ and $(f_1, f_2) \circ A \equiv \{(f_1, f_2) \circ a : a \in A\}$.
showed that when one replaces WPO by IR in Nash’s theorem, exactly two solutions become admissible: \( N \) and \( D \), where \( D \)—the disagreement solution—is defined by \( D(S,d) \equiv d \). The following axiom is a weakening of IR.

**Disagreement Domination** (DID): There does not exist a problem \((S, d) \in \mathcal{B}\) such that \( \mu(S, d) \not\leq d \).

All the above mentioned axioms have clear economic interpretations;\(^4\) therefore, for the sake of brevity, I will not discuss them.

### 3 The result

**Theorem 1.** There are exactly two bargaining solutions that satisfy independence of equivalent utility representations, independence of irrelevant alternatives, symmetry, and disagreement domination: the Nash solution and the disagreement solution.

To prove the theorem, I make use of two lemmas. For stating (and proving) the lemmas, the following definition and notation are needed. Say that a bargaining problem \((S, d)\) is \(d\)-comprehensive if the following is true for every \(y\): if \(d \leq y \leq x\) for some \(x \in S\) then \(y \in S\). Let \(\mathcal{B}_C \equiv \{(S, d) \in \mathcal{B} : (S, d)\) is \(d\)-comprehensive\}. A solution on \(\mathcal{B}_C\) is a function \(\mu: \mathcal{B}_C \to \mathbb{R}^2\) that satisfies \(\mu(S, d) \in S\) for all \((S, d) \in \mathcal{B}_C\).

**Lemma 1.** Let \(\mu\) be a solution on \(\mathcal{B}_C\). If it satisfies independence of equivalent utility representations, independence of irrelevant alternatives, symmetry, and disagreement domination, then it is individually rational.

**Proof.** Let \(\mu\) be a solution on \(\mathcal{B}_C\) that satisfies the aforementioned axioms. Assume by contradiction that there is a problem \((S, d) \in \mathcal{B}_C\) such that \((a, b) \equiv \mu(S, d) \notin S_d\).

\(^4\)See, e.g., Thomson (1994). The only axiom from above which is new to the literature is DID; this axiom’s requirement, in turn, is straightforward and (intuitively speaking) very weak.
By DID and IEUR we can assume, wlog, that $d = 0$ and $a < 0 < b$. Let $\epsilon > 0$ be such that $x \equiv (\epsilon, 0) \in S$ and consider $V = V(b) \equiv \text{conv}\{0, (a, b), x\}$. By IIA, $\mu(V, d) = (a, b)$. I will now obtain the desired contradiction by showing that $\mu(V, d) = (a, b)$ is impossible. To this end, I will assume that $\epsilon = 1$, which, is view of IEUR, is wlog.

By IEUR, note that if $\mu(V(b), d) = (a, b)$, then $\mu(V(b'), d) = (a, b')$ for all $b' > 0$. Consider $W(b) \equiv \text{conv}\{0, (a, b), (b, a), (1, 0), (0, 1)\}$. By SY, $\mu(W(b), d)$ is on the 45° line and by DID it is weakly above $d = 0$. Look at $b = 1 - a$. For this particular $b$ we have $\frac{a + b}{2} = \frac{1}{2}$. Therefore, $\mu(W(1 - a), d) \in [0; (\frac{1}{2}, \frac{1}{2})]$. To obtain the desired contradiction, I will show that $[0; (\frac{1}{2}, \frac{1}{2})] \subset V(1 - a)$. To show this, in turn, it is enough to establish $(\frac{1}{2}, \frac{1}{2}) \in V(1 - a)$. Note that for every $\theta \in [0, 1]$ we have $\theta(a, 1 - a) + (1 - \theta)(1, 0) = (\theta a + 1 - \theta, \theta(1 - a)) = (\theta(a - 1) + 1, \theta(1 - a)) \in V(1 - a)$. Taking $\theta = \frac{1}{2(1 - a)}$ we obtain $(\frac{1}{2}, \frac{1}{2}) \in V(1 - a)$.

Therefore, since $\mu(W(1 - a), d) \in [0; (\frac{1}{2}, \frac{1}{2})] \subset V(1 - a)$, it follows from IIA that $\mu(V(1 - a), d)$ is on the 45° line, in contradiction to $\mu(V(1 - a), d) = (a, 1 - a)$.

**Lemma 2.** There are exactly two solutions on $\mathcal{B}_C$ that satisfy independence of equivalent utility representations, independence of irrelevant alternatives, symmetry, and disagreement domination: the Nash solution and the disagreement solution.

**Proof.** Both $N$ and $D$ satisfy the axioms listed in Lemma 2. Conversely, let $\mu$ be any solution on $\mathcal{B}_C$ that satisfies the axioms. By the Lemma 1, $\mu$ satisfies IR. Therefore, by Roth (1979), $\mu \in \{N, D\}$. 

With Lemma 2 at hand, proving Theorem 1 boils down to passing from the restricted domain $\mathcal{B}_C$ to the full domain $\mathcal{B}$. To this end, the following notation will be useful.

Given $(S, d) \in \mathcal{B}$, let $Q(S, d) \equiv \{(Q, d) \in \mathcal{B}_C : Q \supset S\}$ and let:

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5The existence of such an $x$ follows from the combination of (i) the assumption that $S_d$ has a non-empty interior, and (ii) $d$-comprehensiveness.

6Given two vectors, $x$ and $y$, the segment connecting them is denoted $[x; y]$.

7Roth’s theorem holds also on the domain $\mathcal{B}_C$. 

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\[ F(S, d) \equiv \cap_{Q \in \mathcal{Q}(S, d)} Q. \]

In words, \((F(S, d), d)\) is the “smallest” \(d\)-comprehensive problem containing \((S, d)\).

**Proof of Theorem 1:** Both \(N\) and \(D\) satisfy the axioms listed in the theorem. Conversely, let \(\mu\) be any solution that satisfies them. Let \((S, d) \in \mathcal{B}\). By Lemma 2, \(\mu(F(S, d), d) \in \{N(F(S, d), d), D(F(S, d), d)\}\).

Suppose first that \(\mu(F(S, d), d) = N(F(S, d), d)\). Since both \(\mu\) and \(N\) satisfy IIA, \(\mu(S, d) = N(S, d)\). In this case, I argue, \(\mu = N\). By Lemma 2, this equality holds on \(\mathcal{B}_C\). Therefore, if it is violated then there is some \((T, e) \in \mathcal{B} \setminus \mathcal{B}_C\) such that \(\mu(T, e) \neq N(T, e)\). Therefore, by IIA, \(\mu(F(T, e), e) \neq N(F(T, e), e)\)—in contradiction to the fact that \(\mu\) coincides with \(N\) on \(\mathcal{B}_C\).\(^8\) The case \(\mu(F(S, d), d) = D(F(S, d), d)\) is covered analogously. \(\square\)

The axioms from Theorem 1 are independent. To see this, let us start by looking at the following three examples.

1. All the axioms but IIA: The **Kalai-Smorodinsky solution**, \(KS\) (due to Kalai and Smorodinsky (1975)). \(KS(S, d) \equiv WP(S) \cap [d; a(S, d)]\), where \(a_i(S, d) \equiv \max\{x_i : x \in S_d\}\).

2. All the axioms but IEUR: The **egalitarian solution**, \(E\) (due to Kalai (1977)). \(E(S, d) \equiv d + (e, e)\), where \(e\) is the maximal number such that this expression is in \(S\).

3. All the axioms but SY: The **dictatorial solution**, \(D_i(S, d) \equiv (a_i(S, d), d_j)\).

An example of a solution that satisfies all the axioms but DID would be, not surprisingly, more pathological. To describe such an example, the following notation

\(^8\)If \(\mu(F(T, e), e) = N(F(T, e), e)\) then by IIA \(\mu(T, e) = N(T, e)\)—in contradiction to our assumption.
would be useful. Given a feasible set $S$, let $b_i(S) = \min\{x_i : x \in S\}$. The point $b(S) = (b_1(S), b_2(S))$ is the “bottom corner” of $S$ (which need not be a point of $S$). Consider the solution $\mu^\ast$, which is defined by $\mu^\ast(S, d) \equiv b(S) + (r, 0)$, where $r$ is the minimal number such that this expression is in $S$. It is easy to verify that $\mu^\ast$ satisfies all the axioms but DID.

Finally, it is worth noting that while there are economically meaningful bargaining solutions that violate IR, no such solution violates DID; furthermore, one may argue that a function $f : \mathcal{B} \to \mathbb{R}^2$ such that $f(S, d) \in S$ for all $(S, d) \in \mathcal{B}$ cannot seriously be thought of as a bargaining solution if it violates DID. Example of a solution that satisfies DID but violates IR is the *utilitarian solution*—the solution that maximizes the sum of the players’ utilities over the feasible set. More generally, any solution that depends on the feasible set but not on the disagreement point will inevitably violate IR. To demonstrate one additional such solution, which has its economic merits despite the fact that it violates IR, the following definition is needed. Given a feasible set $S$, let $A_i(S)$ be the maximum payoff that player $i$ can obtain in $S$, namely $A_i(S) \equiv \max\{x_i : x \in S\}$. The following variant of the *equal loss solution* (Chun (1988)) violates IR: for every $(S, d) \in \mathcal{B}$ the solution picks the point $(A_1(S) - l, A_2(S) - l)$, where $l$ is the minimal number such that this expression is in $S$.

References


Kalai, E., (1977), Proportional solutions to bargaining situations: interpersonal util-

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9Strictly speaking, to make it a “solution” in our formal sense, one needs to defined a selection from the utilitarian correspondence, as the maximizer of the utility sum may not be unique.


