Bribing in first-price auctions

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Abstract

I study a symmetric 2-bidder IPV first-price auction prior to which one designated bidder can offer his rival a bribe in exchange for the latter’s departure from the auction. I focus on pure and undominated strategies, and on continuous monotonic equilibria—equilibria in which the bribing function is continuous and nondecreasing. When types are distributed continuously on the unit interval, such an equilibrium may not exist. Moreover, if it does exist, then it is necessarily trivial—its bribing function is identically zero. I provide a sufficient condition for the existence of a trivial equilibrium. When types are distributed continuously on an interval whose minimum is strictly positive, a non-trivial (continuous, monotonic) equilibrium may exist, but it must be pooling—a common offer is made, independent of the briber’s type. I provide a sufficient condition for the existence of such an equilibrium. When types are distributed continuously on the unit interval and dominated (pure) strategies are allowed, a non-trivial non-pooling (continuous, monotonic) equilibrium exists, at least under the uniform type distribution.

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1 Introduction

Bidder collusion is a well-documented problem.\footnote{See, for example, Baldwin et al. (1997), Cassady (1967), and Porter and Zona (1993).} The theoretical literature typically models collusion as a pre-auction one-shot interaction among the cartel members. More specifically, collusion is typically modeled either as a revelation game played by the cartel members, in many cases with the aid of an incentiveless third party, or as a “knockout auction”—an auction that the cartel members run among themselves, for the right to participate in the real auction.\footnote{Leading examples include Graham and Marshall (1987), Mailath and Zemsky (1991), Marshall and Marx (2007), and McAfee and McMillan (1992).}

This approach suffers from two drawbacks. First of all, it models collusion as a static affair. As such, it misses the important issue of signaling, which presents itself in sequential interactions: during the formation of the collusive agreement, the bidders’ moves are indirect signals of their private information, signals that give rise to adverse selection. The second (and related) drawback is that bidders’ negotiations seems to be more naturally modeled as simple bargaining protocols, and not, for example, as abstract revelation mechanisms.

Both drawbacks were addressed by Esö and Schummer (2004, henceforth ES), who studied the following extensive form game: two players are about to attend a symmetric second-price auction with independent and private values (henceforth IPV), and prior to the auction one fixed player has an opportunity to offer the other player a bribe in exchange for the latter’s abstention. If the bribe is accepted then the briber becomes the sole participant in the auction and obtains the good for free (the reserve price is zero), while if it is rejected then the pre-auction stage ends and both players go on to compete in the auction noncooperatively.

When studying collusion in simple sequential mechanisms, the ES game is the right place to begin—the second-price format is one of the best-known and most-studied auction formats, and the “take-it-or-leave-it” (henceforth TIOLI) protocol is
the simplest bargaining protocol. However, one would like to take the analysis further: examination of alternative models will shed light on the driving forces behind the ES results, and will help clarify how these results depend on the specific details of the ES game. This leads me to study the same game as ES with one exception—instead of second-price, I consider a first-price auction. As we shall see, there are dramatic differences in the predictions of the two models.

The rest of the paper is organized as follows. Subsection 1.1 outlines the results and subsection 1.2 overviews the related literature. Section 2 lays down the formal model, Section 3 contains the results, Section 4 concludes, and the Appendix collects proofs which are omitted from the text.

1.1 Outline of the results

ES study the second-price game under the assumption that the players employ pure strategies and, moreover, they employ their weakly dominant auction-strategies and bid truthfully in all auctions. Assuming a differentiable, strictly increasing, log-concave distribution of valuations, they prove that this game has a unique perfect Bayesian equilibrium (PBE) that involves bribes and in which the bribing function is continuous; moreover, they derive the closed-form description of this equilibrium. The bribing function of this equilibrium—henceforth, the $ES$ equilibrium—starts at the origin, it is strictly increasing up to a certain point, and then it becomes flat.

I also take PBE to be the solution concept, and, similarly to ES, I restrict my attention to pure strategies. In most of the paper I assume that the players employ

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3The players face only one auction, but it can be played following different bribing-stage histories. Auctions that follow different bribing-stage histories are, formally speaking, different continuation games.

4When player 1 makes the first move in the game and offers a bribe, this bribe is a function of his valuation (type), hence his behavior is given by a bribing function.

5The bribing function’s exact shape depends on the bidders’ type distribution. For example, in the case of the uniform distribution, it is piecewise linear.
undominated bidding strategies in the auction.\textsuperscript{6}

Under the assumptions of pure strategies and undominated bidding, the following key point obtains: \textbf{if the equilibrium bribing function is continuous and nondecreasing on some interval, then it is constant on that interval.} This implies that when the minimal type is zero, any equilibrium in which the bribing function is continuous and nondecreasing (a \textit{continuous monotonic equilibrium}) is \textit{trivial}—the equilibrium bribing function is identically zero. I obtain (in Theorem 2) a sufficient condition for the existence of a trivial equilibrium. I also obtain (in Theorems 3 and 4) sufficient conditions for its nonexistence.\textsuperscript{7} These results are for the case where types are distributed continuously on the unit interval.

I also consider a more general type space, an arbitrary compact interval in $\mathbb{R}_+$. The gist of the aforementioned key point—that a continuous nondecreasing equilibrium bribing function must be constant—still has a bite, but this bite is limited. Now, though all briber types must offer the same bribe, this common offer need not be zero. Continuity and monotonicity still imply pooling, but not necessarily triviality.

The driving force behind the aforementioned results stems from the informational link between the bribing stage and the bidding stage: information which is signaled through the bribe influences bidding in the auction that follows the rejection of that bribe. Under the assumptions of pure strategies and undominated bidding, this influence precludes incentive provision. In particular, the only possibility for a continuous and monotonic equilibrium to exist in this case is that it be completely pooling. In this regard, the ES second-price game is special: it is a special feature of the second-price format that bidding in the auction is independent of any information that is inferred from the pre-auction activity.

The aforementioned informational separation in the ES game, which is due to the fact that ES assume truthful bidding in the auction, has two significant implications. First, the bribing function in every equilibrium of the second-price game must be

\textsuperscript{6}I relax the restriction of undominated bidding in subsection 3.4 (and only there).

\textsuperscript{7}This nonexistence is under the pure-strategies assumption.
nondecreasing; second, every bribery-involving equilibrium of the second-price game leads to inefficiency with a positive probability. None of these implications is in place under other bidding behaviors. Though my focus is on the first-price game, I briefly revisit the second-price game of ES, without imposing truthful bidding. I construct an equilibrium of this game in which the bribing function is nonmonotonic and under which the allocation of the good is ex post efficient.

After having dealt with general type intervals and having revisited the second-price game, I go back to the first-price game in which types are distributed on the unit interval. This time, however, I do not impose undominated bidding. This parallels the relaxation of truthful bidding in the second-price game. Under the assumption of a uniform type distribution, I construct an equilibrium which, qualitatively speaking, looks like the ES equilibrium of the second-price game: the bribing function starts at the origin, it is strictly increasing at first, and then it becomes flat. Thus, under both the first- and second-price formats the possibility of dominated behavior at the auction stage brings an improvement: in the former case it makes nontrivial equilibrium possible, and in the latter case it makes efficiency possible.

The focus in the paper is on continuous monotonic equilibria. Though continuity is certainly not pathological, it may seem a little restrictive, because discontinuous equilibria—e.g., equilibria involving step functions—sometimes arise naturally in games with adverse selection and a continuum of types. I therefore examine the relaxation of continuity to restricted discontinuity. An equilibrium is restrictedly discontinuous if its bribing function has at most finitely many discontinuity points. Even under the weaker notion of restricted discontinuity, equilibrium nonexistence raises its ugly head: given a certain distributional condition, which is satisfied by a large family of distributions, there does not exist a restrictedly discontinuous equilibrium.

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8See, for example, Leininger et al. (1989).
1.2 Related literature

Other than the paper of ES, there are two papers in the literature that consider TIOILI bribing. Chen and Tauman (2006, henceforth CT) consider such collusion when the auction format is second-price and Kivetz and Tauman (2010, henceforth KT) consider such collusion when the auction format is first-price. Both models differ substantially from the one considered here. In the paper of CT the environment is such that besides the two colluding bidders (the cartel members) there is a random population of other bidders, from which the cartel members can hire shill bidders. Additionally, this paper assumes that each bidder has either two or three types, as opposed to the continuum-of-types setting, which is considered here and in the paper of ES. In the paper of KT, types are commonly known among the bidders; in particular, the game which is analyzed in this paper is not a signaling game.

In a recent paper, Lopomo et al. (2011) consider collusion in first-price auctions, but in the spirit of mechanism design (as opposed to TIOILI bribing). They prove the impossibility of profitable collusion in first-price IPV auctions in the following sense: when the cartel cannot control the bids of its members, bids are discrete, and there are two symmetric bidders with two possible valuations, then the best that the cartel can achieve converges to the noncooperative payoff as the bid increment becomes smaller.\(^9\) Similarly to the papers that were mentioned in footnote 2, collusion in this paper is signaling-free.

\(^9\)They also carry out numerical analysis that suggests that the result continues to hold in more general environments. More specifically, they consider the possibility of (i) two nonsymmetric bidders each of whom has two types, and (ii) two symmetric bidders with \(n\) types.
2 Model

2.1 The game

There are two risk-neutral expected-utility-maximizing players, player 1 and player 2, who are about to attend a first-price auction for a single indivisible good. Whenever player \( i \) and player \( j \) are mentioned in the same sentence, it is implicitly assumed that \( j \neq i \). The reserve price is zero.\(^{10}\) Player \( i \)'s valuation for the good is a random draw from \( F \), which is an atomless full-support distribution function on \([0, 1] \).\(^{11}\) The valuations (or types) are independent, are denoted by \( \theta_i \), and are private information.

Prior to the auction player 1 makes a TIOLI bribe offer to player 2 in exchange for player 2 leaving the auction (in which case player 1 obtains the good for free). He can offer any nonnegative amount, but can only make one offer. Not offering a bribe is a feasible action, which is modeled as “offering zero.” If the offer is accepted, then player 2, who has the ability to commit, departs from the auction. If, on the other hand, it is rejected, then the pre-auction stage ends, and both players go on to compete noncooperatively in the auction. Call this game the first-price TIOLI game.

A (pure) strategy for player 1 consists of a bribing function and a family of bidding functions. The former assigns a nonnegative bribe for every type and the latter prescribes bidding behavior in every auction (that is, in every continuation game). For player 2, a (pure) strategy consists of an acceptance rule and a family of bidding functions. The former prescribes an “accept/reject” decision for every bribe offered by player 1, as a function of player 2’s type.\(^{12}\) I restrict my attention to pure strategies.\(^{13}\)

The first-price TIOLI game is a signaling game with a continuum of types. In such games, equilibria are typically insensitive to the behavior of a single (bordering) type.

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\(^{10}\)The extent to which this assumption can be relaxed will be discussed in subsection 3.2.

\(^{11}\)In subsection 3.3 I will consider a more general type space, \([\theta, \theta] \subset \mathbb{R}_+\).

\(^{12}\)It is implicit in the definition of a strategy that all the functions under consideration are Borel measurable.

\(^{13}\)In particular, bidding in the auction is pure.
For example, suppose that there is an equilibrium in which all types $\theta_2 < \frac{1}{2}$ accept a certain bribe $b^*$, and all types $\theta_2 > \frac{1}{2}$ reject it. Then the threshold type $\theta_2 = \frac{1}{2}$ is indifferent between these two actions; either action can therefore be supported in equilibrium. Throughout the paper, any statement of the form “$\sigma$ is the unique equilibrium profile such that...” means that it is unique up to the behavior of such bordering types.

The first-price TIOLI game can give rise to auctions in which $\theta_1$ is commonly known. To see how, consider the case where player 1’s bribing function is invertible on some subinterval of its domain, and his type, $\theta_1$, belongs to that subinterval; if his bribe is rejected, then the auction that follows is such that $\theta_1$ is common knowledge. In the Bayesian Nash Equilibrium (BNE) of this auction, all the types of the informed player (player 2) submit the “minimally winning” bid if they find winning worthwhile. In terms of modeling, this is made possible by allowing the informed player to submit, in addition to the regular bids, bids of the form $r^+$, for every $r \in \mathbb{R}_+$. A bid $r^+$ wins with certainty against any $r' \leq r$ and loses against any $r' > r$. When a player wins with the bid $r^+$, he pays $r$.\footnote{In a repeated first-price auction model, Blume and Heidhues (2006) consider the possibility of submitting a special bid, $0^+$, which is identical to 0 except that it wins for sure if the highest competing bid is 0. Rachmilevitch (2011) follows Blume and Heidhues and also allows for the special bid $0^+$ in a repeated auction model. The set \{r$^+$|$r \in \mathbb{R}_+$\} generalizes $0^+$.}

Finally, the following piece of notation will be useful. Let $\pi(\theta)$ denote the expected payoff of type $\theta$ in the symmetric BNE of the noncooperative symmetric first-price auction which is played under the a priori information $F$ (it is well known that this BNE exists, and is unique). This payoff satisfies $\pi(\theta) = \int_0^\theta F(t)dt$.\footnote{One way to see this is by invoking the well-known payoff equivalence result (Myerson (1981)), which implies that the expected payoff of type $\theta$ in the symmetric equilibrium of the first-price auction is the same as his expected payoff in the symmetric (dominant strategy) equilibrium of the second-price auction. The latter payoff is $\int_0^\theta (\theta - t)f(t)dt = \int_0^\theta F(t)dt$.}
2.2 Solution concept

Recall that a perfect Bayesian equilibrium (PBE) is a strategy-belief pair with the property that each player is playing a best-response (given his beliefs) whenever he is called to make a move, and beliefs satisfy Bayes’ rule whenever possible. In the sequel, an equilibrium (of the first-price TIOLI game) means a PBE in which the players do not employ dominated bidding strategies in any on-path auction. The undominated-bidding assumption will be relaxed in subsection 3.4, and only there.\footnote{In subsection 3.4, “equilibrium” simply means PBE.}

Our game is a multi-stage signaling game, and therefore involves two kinds of beliefs. First, there are the beliefs about the opponent’s type: once player $i$ makes a move, player $j$ updates his beliefs about $\theta_i$. Additionally, there are beliefs about the opponent’s unobservable moves in the extensive form.\footnote{I do not introduce the formalism for systems of beliefs—as collections of probability distributions—because it will not be needed in the sequel.}

Given a strategy profile $\sigma$ and a number $b \geq 0$, let $T(b|\sigma)$ be the set of briber types who offer the bribe $b$ when $\sigma$ is played. There are four possibilities:

1. $T(b|\sigma)$ is empty,
2. $T(b|\sigma)$ is a singleton,
3. $T(b|\sigma)$ contains more than one element, but has a zero Lebesgue measure, and
4. $T(b|\sigma)$ has a strictly positive Lebesgue measure.

These four possibilities impose different restrictions on beliefs of the first kind, as follows. Suppose that the equilibrium strategy is $\sigma$ and player 1 offered $b$. Having seen $b$, player 2 updates his belief about $\theta_1$. If $T \equiv T(b|\sigma)$ has a positive measure, then Bayes’ rule applies, and player 2 uses it to update his belief. If $T$ is empty then $b$ is a signal of a sure deviation, and player 2’s belief about $\theta_1$ is unrestricted. If $T$ is a singleton, then Bayes’ rule does not apply (the denominator in Bayes’ formula
is zero), but this, of course, is a mere technicality: player 2 learns player 1’s type perfectly in this case, and his belief is updated accordingly. Finally, in case 3 (like in cases 1 and 2) Bayes’ rule does not apply, which implies that player 2 is free to update his belief in any way. It seems reasonable to impose a restriction analogous to the one from case 2, and demand that any such belief put positive probability mass only on \( T \); though this restriction is natural it will not play a role in any of the results, and I will therefore not impose it. Similarly to belief-update on player 2’s part, bribe-rejection leads player 1 to update his belief about \( \theta_2 \) using Bayes’ rule, if it applies; following a bribe-rejection which is a signal of a sure deviation by player 2 (or that happens on the path but only with zero probability), player 1 is free to adopt any belief about \( \theta_2 \).

Beliefs of the second kind involve a bit more subtlety. Consider the case where the auction is actually reached. This auction is a continuation game in which the players move simultaneously. From the standpoint of each player \( i \), however, this continuation game looks like an extensive form game in which the opponent, player \( j \), makes an unobservable move (submits a bid) prior to \( i \)’s move. In such an extensive form, beliefs are specified for player \( i \) regarding \( j \)’s unobservable move. In Theorem 2 and Theorem 7 I assume such an extensive form, where player 1 is the “first mover” in the auction who tenders his bid before player 2 does (player 2, of course, does not see that bid). This is important for the specification of the beliefs that are needed in order to support the equilibria which are constructed in the proofs of these theorems. This subtlety does not arise in the other theorems.

An equilibrium is **continuous** if its bribing function is continuous. It is **restrictedly discontinuous** if its bribing function has at most finitely many discontinuity points. It is **monotonic** if its bribing function is nondecreasing. It is **trivial** if its bribing function is identically zero.
3 Results

Lemma 1. Let $b$ be the bribing function of a monotonic equilibrium of the first-price TIOLI game and let $J \subset [0, 1]$ be a nondegenerate interval on which $b$ is continuous. Then $b$ is constant on $J$.

The intuition behind the lemma is as follows. If the bribing function is continuous and nondecreasing, and if, in addition, it is nonconstant, then there exists a nondegenerate interval on which it is strictly increasing. Therefore, there is a continuum of types who reveal themselves through their bribes. When such a bribe is rejected in equilibrium, the continuation auction is such that $\theta_1$ is common knowledge but $\theta_2$ is not. In the unique BNE of such an auction, player 1 submits the bid $\theta_1$ and player 2 bids $\theta_1^+$ and wins. This follows from the undominated-bidding assumption, which is crucial. Therefore, when player 1 of type $\theta_1$ makes a nondetectable deviation and mimics a nearby type $x < \theta_1$—where $x$ is also a type who is supposed to reveal himself in equilibrium—this deviation tricks player 2 to submitting a low bid in the post-rejection auction (namely, $x^+$). This “tricking” makes any continuous monotonic nontrivial equilibrium unravel.

More specifically, mimicking type $x < \theta_1$ has three implications:

- (I) It decreases the probability that the bribe will be accepted,

- (II) it makes acceptance more attractive, because the bribe is lower, and

- (III) it allows the briber to obtain the good following the bribe’s rejection at a price that he is willing to pay: $x < \theta_1$.

Implications (I) and (II) present an obvious tradeoff. It is the addition of (III) that makes a bribery-involving equilibrium unravel.

In fact, if there is a sufficiently low type $x$ who is supposed to reveal himself, then (III) reverses the effect of implication (I) in player 1’s strategic reasoning, in the following sense. Holding player 1’s expected auction-payoff fixed, the cost-benefit
analysis that (I) and (II) present is obvious: a smaller bribe is less likely to be accepted, but the event of its acceptance is more attractive. This tradeoff has, informally speaking, an “interior solution.” Taking (III) into account, however, gives rise to a “corner solution,” in the following sense: when player 1 of type $\theta_1 > 0$ mimics a small type $x$ by offering the (small) bribe $b(x)$, this bribe will likely be rejected by player 2, who, consequently, will bid slightly above $x$ in the auction; then, player 1 can bid slightly above player 2 and obtain the good. This move is clearly profitable if $x < b(\theta_1)$. This consideration incentivizes player 1 to mimic the lowest type who is supposed to reveal himself—a “corner solution.” Furthermore, note that if there are arbitrarily small types who are supposed to reveal themselves—i.e., if $b$ is strictly increasing near zero—then player 1 has an incentive to mimic types arbitrarily close to zero, so clearly $b$ cannot be sustained in equilibrium.

Before we proceed to the first theorem, it is worth noting that the aforementioned (informal) explanation implies that the claim of Lemma 1 follows from the fact that monotonicity and continuity of a candidate equilibrium-bribing-function imply that it must be strictly increasing near the origin. That the combination of monotonicity and continuity implies strict monotonicity near the origin is easy to see: suppose, contrawise, that $b$ is nondecreasing and monotonic, and that it is not strictly increasing near zero; then it is zero on an interval $[0, a]$, for some $0 < a < 1$, and strictly increasing on $(a, a + \delta)$ for a sufficiently small $\delta > 0$. The threshold type, $a$, must weakly prefer his equilibrium payoff, $\pi(a)$, to the one he can obtain by mimicking type $a + \epsilon$, for any $\epsilon \in (0, \delta)$. This implies

$$\pi(a) \geq F(a)a. \tag{1}$$

The reason is that all $\theta_2 \leq a + \epsilon$ accept the bribe $b(a + \epsilon)$, hence type $a$ can

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\[18\] Consider an on-path auction that follows the rejection of the revealing offer $b(a + \epsilon)$. In such an auction player 1 submits a bid which equals his type (for a formal proof, see Step 2 in the proof of Lemma 1 in the Appendix).
guarantee to himself at least $F(a+\epsilon)[a-b(a+\epsilon)]$, and he can do this for arbitrarily small $\epsilon$’s. However, (1) is impossible, since $\pi(a) = F(a)(a-t(a))$, where $t(a)$, the price paid conditional on winning in the noncooperative equilibrium, is strictly positive for $a > 0$.

Based on Lemma 1, the first main result can be stated and proved easily.

**Theorem 1.** *Every continuous monotonic equilibrium of the first-price TIOLI game is trivial.*

**Proof.** Consider a continuous monotonic equilibrium. Let $b$ be its bribing function. By Lemma 1, $b$ is constant. Obviously, $b(0) = 0$. Therefore $b \equiv 0$. \qed

Theorem 1 says that every continuous monotonic equilibrium must be trivial, but does not say anything regarding its existence. Next, Theorem 2 identifies conditions under which such existence obtains. Before we turn to this theorem, a brief informal explanation of the idea behind its proof will be helpful.

Since a continuous monotonic equilibrium must be trivial, player 1 should not have an incentive to deviate to a strictly positive bribe. Let $G(b)$ be the probability that the off-path bribe $b > 0$ is accepted.\(^{19}\) The associated payoff for player 1 from the deviation to $b$ is therefore $G(b)(\theta_1 - b) + (1-G(b))\phi$, where $\phi$ is a shorthand for the payoff he expects in the post-rejection-of-$b$ auction. To sustain equilibrium, we must have

$$G(b)(\theta_1 - b) + (1-G(b))\phi \leq \pi(\theta_1).$$

(2)

In order to achieve this, I construct equilibrium strategies the purpose of which is to make the LHS of (2) as small as possible. I start by making $\phi$ as small as possible, which, in turn, corresponds to assigning player 2 the most aggressive bidding permitted by PBE. Since player 2’s response to the offer $b > 0$ must be optimal for him, he cannot bid more than $(\theta_2 - b)$; I let him submit this bid.\(^{20}\) Additionally, I equip

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\(^{19}\)Note that $G \geq F$, because all $\theta_2 < b$ accept $b$.

\(^{20}\)More precisely, I let him bid $(\theta_2 - b)^+$. 
him with the most aggressive acceptance policy, so \( G = F \). The following theorem identifies conditions under which this behavior can be sustained in equilibrium.

**Theorem 2.** Suppose that \( F \) is differentiable and that it satisfies \( 2F(t) + tf(t) \geq 1 \) for all \( t \in (0,1] \), where \( f = F' \). Then, the first-price TIOLI game has a trivial equilibrium.

**Proof.** Let \( F \) be a distribution that satisfies the above-mentioned condition. Consider the following strategy.

Player 1 is instructed to offer zero independent of his type and after the rejection of zero, he is instructed to bid as in the one-shot symmetric BNE of the auction. Player 2 is instructed to reject zero and bid as in the symmetric BNE following this rejection, and is instructed to accept \( b > 0 \) if and only if \( \theta_2 \leq b \). Following the rejection of \( b \), player 2 believes that player 1 is bidding \( \theta_2 - b \), and he therefore bids \( (\theta_2 - b)^+ \);\(^{21}\) player 1 is prescribed his optimal bid in this post-rejection-of-\( b \) information set.\(^{22}\)

We need to verify that the aforementioned bribing policy is ex ante optimal for player 1 and that the aforementioned behavior is optimal for player 2.

**Optimality for player 2:** It is obviously optimal to accept \( b \geq \theta_2 \). Consider then \( b \in (0, \theta_2) \). In this case, the belief that player 1 bids \( \theta_2 - b \) clearly supports rejection and bidding \( (\theta_2 - b)^+ \) as optimal.

**Optimality for player 1:** Consider a deviation to a bribe \( b > 0 \). Clearly we can assume that \( b \leq 1 \). The payoff corresponding to such a deviation is bounded from above by\(^{23}\)

\[^{21}\text{Note the special feature that player 2's believes depend on his own type. This is a little strange, but, nevertheless, allowed by PBE.}\]
\[^{22}\text{It is clear that given player 2's bidding, a best-response exists for player 1.}\]
\[^{23}\text{To see that this is an upper bound on player 1's payoff, note that when type } \theta_2 = t > b \text{ rejects the offer } b \text{ he submits the bid } (t - b)^+ \text{ in the auction. The ideal situation for player 1 would be to be informed, once } b \text{ was rejected, what is the rejector's type (namely, what is } t \text{); conditional on this additional information, player 1’s payoff is bounded from above by } \max\{0, \theta_1 - t + b\}.\]

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\[ \Pi(b|\theta_1) \equiv F(b)(\theta_1 - b) + \int_b^1 \max\{0, \theta_1 - t + b\} f(t) dt = \]
\[ = F(b)(\theta_1 - b) + \int_b^{\min\{\theta_1 + b, 1\}} (\theta_1 - t + b) f(t) dt. \]

Suppose first that \( \min\{\theta_1 + b, 1\} = \theta_1 + b \). In this case, the aforementioned payoff is

\[ F(b)(\theta_1 - b) + \int_b^{\theta_1 + b} (\theta_1 - t + b) f(t) dt = \]
\[ = -2F(b)b + F(\theta_1 + b)(\theta_1 + b) - \int_b^{\theta_1 + b} tf(t) dt, \]

and its derivative with respect to \( b \) is \(-f(b)b - 2F(b) + F(b + \theta_1)\), which, by assumption on \( F \), is nonpositive.

Suppose next that \( \min\{\theta_1 + b, 1\} = 1 \). In this case player 1’s expected payoff is (bounded from above by) \( F(b)(\theta_1 - b) + \int_b^1 (\theta_1 - t + b) f(t) dt. \) The derivative of this expression with respect to \( b \) is \( 1 - 2F(b) - f(b)b \leq 0 \), where the inequality is by assumption on \( F \).

Look at the upper bound on the deviator’s payoff, \( \Pi(b|\theta_1) \). This function satisfies the following properties: (i) it is continuous in its argument, (ii) \( \Pi(0|\theta_1) = \pi(\theta_1) \), (iii) it is weakly decreasing on \([0, 1 - \theta_1]\), and (iv) it is weakly decreasing on \([1 - \theta_1, 1]\).

Therefore, there does not exist a profitable deviation. \( \square \)

Intuitively, the sufficient condition in Theorem 2 is that “a lot” of the probability mass is concentrated near zero. In such a case, player 1 is facing, in probabilistic terms, a weak opponent, and is therefore not willing to pay “even a penny” for eliminating him from the auction. There is no shortage of probability distributions that satisfy this condition. For example, \( F(t) = -2t^2 \log t + t^2 \) does. Informally speaking, a distribution function that satisfies this condition is “very concave” near the origin. It turns out that if the expected type is at least one half, then the “opposite” of this property—namely, local convexity of the distribution function near the origin—is
sufficient for equilibrium nonexistence. The nonexistence in this case is rather strong: no monotonicity or continuity are required, only restricted discontinuity.

**Theorem 3.** Suppose that the distribution $F$ is twice differentiable, that its derivative $f$ is strictly positive on $(0, 1)$, that it satisfies $F''(0) = f'(0) > 0$, and that $\mathbb{E}(\theta) \geq \frac{1}{2}$. Then, the first-price TIOLI game does not have a restrictedly discontinuous equilibrium.

The intuition for why local convexity of $F$ at the origin makes the trivial equilibrium unravel can be seen when considering a deviation to a small positive bribe $\epsilon > 0$, a bribe which is accepted by all $\theta_2 < \epsilon$. If $F$ is convex near zero, then the expected type in $[0, \epsilon)$ is relatively large (i.e., close the right end of this interval). Therefore, for a small positive bribe, player 1 can exclude from the auction with certainty an opponent who is, in expectation, relatively strong (relatively to the bribe).

If $\mathbb{E}(\theta) \geq \frac{1}{2}$ is strengthened to a strict inequality, then nonexistence of a monotonic continuous equilibrium obtains even without local convexity of $F$ at the origin.

**Theorem 4.** Suppose that $\mathbb{E}(\theta) > \frac{1}{2}$. Then, the first-price TIOLI game does not have a continuous monotonic equilibrium.

**Proof.** Make the assumption of the theorem and assume by contradiction that a monotonic continuous equilibrium exists. By Theorem 1, it must be trivial. Pick an $e \in (\frac{1}{2}, \mathbb{E}(\theta))$. Consider player 1 of type $\theta_1 = 1$. When he deviates and offers $e$, all $\theta_2 \leq q$ accept this bribe, where $q \geq e$. If $q = 1$ then this deviation is profitable, because it gives the payoff $1 - e > 1 - \mathbb{E}(\theta)$, and the latter is player 1’s equilibrium payoff (this follows from payoff equivalence). Suppose, on the other hand, that $q < 1$. Since any $\theta_2$ who rejects $e$ does not bid more than $\theta_2 - e$ in the auction, player 1 can win with certainty in the post-rejection auction by bidding $1 - e$. Therefore, player 1 of type $\theta_1 = 1$ can guarantee to himself the payoff $h(q) = F(q)(1 - e) + (1 - F(q))e$. Note that $e > \frac{1}{2}$ implies that $h(q) \geq h(1)$. Therefore, player 1 can guarantee to himself $h(q) \geq h(1) = 1 - e > 1 - \mathbb{E}(\theta)$. \qed
3.1 The link between the bribing stage and the bidding stage

The fact that under the restriction to continuous and nondecreasing bribing functions only trivial equilibria are possible stands in sharp contrast to the existence of the ES equilibrium. The reason for this difference is that in the second-price game, when the players bid truthfully in all auctions, there is complete separation between the bribing stage and the bidding stage: once a player finds himself participating in the auction, it is weakly dominant for him to bid truthfully; the information that was inferred from the bribing stage plays no rule in this decision. When other bidding behavior is permitted, this separation is no longer in effect.

Linking the bidding stage to the bribing stage by allowing history-dependent bidding functions expands the set of equilibria (PBE) of the second-price TIOLI game. Under truthful bidding every equilibrium bribing function must be nondecreasing, and inefficiency is inevitable in a bribery-involving equilibrium; however, when weakly dominated bidding is allowed, bribery-involving equilibria need not be inefficient, and the equilibrium bribing function need not be monotonic.

**Proposition 1.** When nontruthful bidding in the auction is permitted, the second-price TIOLI game has a bribery-involving equilibrium that leads to an ex post efficient allocation, and in which the bribing function is not monotonic.

The equilibrium which is constructed in the proof of Proposition 1 is as follows. Every type \( \theta_1 < 1 \) offers the expectation of player 2’s type conditional on player 2 being weaker; that is, \( \theta_1 < 1 \) offers \( \tilde{b}(\theta_1) \equiv \mathbb{E}(\theta_2|\theta_2 \leq \theta_1) \), which is exactly the price he would have paid (in expectation) in the noncooperative auction, had he won. The maximal type, \( \theta_1 = 1 \), “pools together” with the minimal type, and offers nothing.

The function \( \tilde{b} \) is invertible, hence player 2 learns player 1’s type perfectly once seeing a strictly positive offer. He is instructed to accept the bribe if and only if

\[ \text{This inefficiency is easy to see: when type } \theta_1 \text{ reveals himself through } b(\theta_1) > 0, \text{ all } \theta_2 \in (\theta_1, \theta_1 + b(\theta_1)) \text{ accept his offer and drop out.} \]
\( \theta_2 \leq \theta_1 \) and he bids truthfully in the auction. Taking this as given, player 1 knows that once he offered according to \( \tilde{b} \) and his offer has been rejected, he will compete in the auction against a stronger opponent who bids truthfully; his own bidding is therefore immaterial for him in this event (as long as he does not bid more than his valuation), which makes the bid \( \theta_1 - \tilde{b}(\theta_1) \) optimal for him; this bid, in turn, is exactly the bid that rationalizes player 2’s efficient acceptance rule as a best-response. Note that in this equilibrium player 1’s expected payoff equals his noncooperative payoff, \( \pi(\theta_1) \). The minimal and maximal types, \( \theta_1 \in \{0, 1\} \), obtain this payoff in the auction, while all other types obtain it through bribing.

### 3.2 Positive reserve price

Some of the aforementioned results continue to hold if the reserve price is not zero, but some positive number \( r \in (0, 1) \). Lemma 1 continues to hold.\(^{25}\) The following counterpart of Theorem 1 therefore obtains.

**Theorem 5.** Consider the first-price TIOLI game with a reserve price \( r \in (0, 1) \). Then, every monotonic equilibrium in which the bribing function is continuous on \((r, 1]\) is trivial.

**Proof.** Let \( r \in (0, 1) \) be the reserve price and consider an arbitrary equilibrium in which the bribing function, \( b \), is continuous on \((r, 1]\). By Lemma 1 it is constant on this interval, taking the value \( B \geq 0 \). Obviously, \( b(\theta_1) = 0 \) for all \( \theta_1 \leq r \). Assume by contradiction that \( B > 0 \). Note the the threshold type \( \theta_1 = r \) is indifferent between offering zero and offering \( B \).\(^{26}\) The probability that \( B \) is accepted is strictly positive (all \( \theta_2 < \max\{r, B\} \) accept \( B \)), and conditional on this event the payoff of \( \theta_1 = r \)

\(^{25}\)It is readily verified that the arguments in its proof have nothing to do with the reserve price.

\(^{26}\)It is obvious that he weakly prefers the bribe zero, because \( B \) gives him a nonpositive payoff (independent of player 2’s response); if, however, he strictly prefers the bribe zero, this means that types of the form \( \theta_1 = r + \epsilon \), where \( \epsilon > 0 \) is small, also strictly prefer zero, in contradiction to equilibrium.
is strictly negative; therefore, it must be that \( B \) is rejected with a strictly positive probability on the path, and that the payoff for \( \theta_1 = r \) in this continuation auction is strictly positive, which is, of course, impossible.

Theorem 2 and Theorem 3 refer to local conditions at the origin, where the reserve price coincides with the minimal type; hence, they do not have counterparts in a model with a positive reserve price. Theorem 4, however, does not build on local conditions, and can be extended as to accommodate a positive reserve price.

**Theorem 6.** Suppose that \( \mathbb{E}(\theta) > \frac{1}{2} \). Then, the first-price TIOLI game with a reserve price \( r \in (0, 1) \) does not have a continuous monotonic equilibrium.

**Sketch of Proof:** Assume by contradiction that a monotonic continuous equilibrium exists. Then, by Theorem 5, it must be trivial. Like in the proof of Theorem 4, consider a deviation by \( \theta_1 = 1 \) to the bribe \( e \in (\frac{1}{2}, \mathbb{E}(\theta)) \), which is accepted by all \( \theta_2 \leq q \). If \( q = 1 \) then the payoff from this deviation is \( 1 - e > 1 - \mathbb{E}(\theta) \). Hence, it is enough in this case to verify that the latter expression is weakly greater than player 1’s expected payoff in the (symmetric BNE of the) noncooperative auction. This is indeed the case, because the aforementioned payoff is a decreasing function of \( r \).\(^{27}\) The case \( q < 1 \) is covered by the arguments from the proof of Theorem 4. \( \square \)

### 3.3 Arbitrary type intervals

Consider the model described so far, with one exception—types are drawn from an arbitrary interval, \( \Theta \equiv [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}_+ \). The lion’s share of the analysis which was developed up to and including the last subsection applies;\(^{28}\) there is one point, however, which is different—now, bribing can occur (under certain conditions) in equilibrium.

The main insight from the previous analysis is still in place: if an equilibrium

\(^{27}\)See, for example, Krishna (2002), p.24.

\(^{28}\)The counterparts of Theorem 2 and Theorem 3 hold, when appropriate local conditions are imposed on the behavior of the distribution in a neighborhood of the minimal type.
The bribing function is nondecreasing and continuous on some nondegenerate subinterval of $\Theta$, then it must be constant there. However, as opposed to the special case where the minimal type is zero, in the general case it may be possible for the minimal type to offer a bribe in equilibrium. The equilibrium still must be pooling, but it need not be trivial.

To see this more formally, let $b^* \in (0, \bar{\theta}]$ denote a candidate for a bribe in a pooling equilibrium, in which all $\theta_1$ offer $b^*$. This bribing can be made part of an equilibrium if, for example, it is profitable for every $\theta_2$ to accept it. Necessary conditions for such behavior to be sustainable in equilibrium are

$$\bar{\theta} \geq b^* \geq \pi(\bar{\theta}). \quad (3)$$

The first inequality guarantees that it is incentive compatible for player 1 to offer the bribe and the second inequality guarantees that it is incentive compatible for player 2 to accept it.

Note that (3) simplifies to

$$\mathbb{E}(\theta) \geq \bar{\theta} - \theta. \quad (4)$$

Condition (4) holds if the minimal type is sufficiently far from zero and if the expected type is sufficiently large (relatively to the distance $\bar{\theta} - \theta$).

The intuition behind this condition is as follows. First of all, the fact that the minimal type must be sufficiently far from zero says that there must be common knowledge that all types have sufficiently large gains from trade. With a lower bound on these gains in place, the “tricking” possibilities for player 1 are limited: he can pretend to be weaker than he actually is, but there is a bound to how weak he can

\[29\] The proof of this fact is analogous to the proof of Lemma 1 and is therefore omitted.

\[30\] The function $\pi$ is well-defined, due to the uniqueness of an equilibrium in the symmetric first-price auction. Moreover, since, by payoff-equivalence, the expected payoff of each type in this auction coincides with this type’s payoff in the dominant strategy equilibrium of the second-price auction, it follows that the $\pi$ does not depend on the auctioneer’s reserve price, as long as it is weakly below $\bar{\theta}$.

\[31\] The fact that it is incentive compatible for $\theta_1 = \bar{\theta}$ to offer the bribe implies that it is incentive compatible for all $\theta_1$ (because the function $\theta_1 - \pi(\theta_1)$ is increasing.)
pretend to be. Secondly, the requirement that $\mathbb{E}(\theta)$ be sufficiently large comes from the fact that player 1 needs to be, in expectation, sufficiently strong: this makes it incentive compatible for player 2 to accept the common bribe in a pooling equilibrium, despite the fact that by doing so he may be dropping out of a competition against a substantially weaker opponent. Under the following minor strengthening of (4), a bribery-involving equilibrium exists.

$$\mathbb{E}(\theta) \geq \bar{\theta} - \theta/2.$$  \hfill (5)

**Theorem 7.** Suppose that types are drawn independently from $F$, a full-support distribution on $\Theta = [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}_+$ that satisfies (5). Then the first-price TIOLI game has a pooling bribery-involving equilibrium.

**Proof.** Make the assumptions of the theorem. Let $b^* \equiv \bar{\theta} - \mathbb{E}(\theta)$. Consider a strategy such that all $\theta_1$ offer $b^*$, and player 2 accepts a bribe $b$ if and only if $b \geq b^*$. In the auction (which is reached only off the path), player 2 bids $\theta_2^+$ and player 1 is playing a best-response against this bidding of player 2.\footnote{It is easy to see that such a best-response for player 1 exists.}

Note that when player 1 makes an offer $b' < b^*$, he expects it to be rejected with certainty, and lead to a first-price auction in which the opponent bids as in the dominant strategy equilibrium of the second-price auction; his expected payoff from this auction, therefore, is smaller than $\pi(\theta_1)$. Therefore, the payoff associated with the deviation $b'$ is bounded by $\pi(\theta_1) \leq \pi(\bar{\theta}) = \bar{\theta} - \mathbb{E}(\theta) \leq \bar{\theta} - b^*$, where the second inequality follows from (5).\footnote{To see this, assume by contradiction that $\bar{\theta} - b^* = \bar{\theta} - \mathbb{E}(\theta) < \bar{\theta} - \mathbb{E}(\theta)$. Rearranging yields $\mathbb{E}(\theta) < \bar{\theta} - \theta/2$, in contradiction to (5).} However, adhering to the equilibrium gives the payoff $\theta_1 - b^* \geq \bar{\theta} - b^*$. Player 2’s behavior is supported as optimal by the off-path belief the player 1 submits the bid $\theta_2$.\footnote{Like in the proof of Theorem 2, here, too, the off-path beliefs of player 2 depend on his type.}

One may speculate that in the case of the general type space, the first-price TIOLI
game may have equilibria in which all the types of player 1 offer the same bribe, but in which this bribe is accepted by player 2 only if $\theta_2$ falls short of a certain threshold; otherwise, there is on-path rejection, which leads to an auction. The question whether such equilibria exist remains, at the present time, open. The difficulty lies in the fact that such equilibria (if they at all exist) involve asymmetric on-path auctions.

### 3.4 Bribery-involving equilibrium with dominated bidding strategies

The results obtained for the first-price TIOLI game up to and including the last subsection rely on assumption that the players do not bid more than their valuation in the auction. As we saw with the second-price TIOLI game of ES in subsection 3.1, information revelation at the bribing stage can make it safe for player 1 to employ a dominated bidding strategy in the auction, once it is reached. Here, I utilize such information revelation, together with dominated bidding, to sustain a bribery-involving equilibrium in the first-price TIOLI game. For reasons of tractability, I restrict my attention to the uniform type distribution on $[0, 1]$.

Let $b$ be the unique solution to the following differential equation that satisfies the boundary condition $b(0) = 0$

$$b'(t) = t - b(t). \quad (6)$$

In the Appendix I prove that $b$ is strictly increasing and convex on the unit interval. Let $\eta > 0$ be a sufficiently small constant, so that

$$e^{-\eta} > \max\{\eta, b'(\eta)\}, \quad (7)$$

and
\[ e^{-\eta x}b(x) + (1 - e^{-\eta x})(e^{-\eta x} - b(x)) \geq b(\eta) \quad \forall x \in \mathcal{D} = [0, \eta]. \]  

Define the function \( k: \mathcal{D} \to \mathbb{R}_+ \) by \( k(x) \equiv e^{-\eta x} - b(x) \).

**Lemma 2.** The function \( k \) is strictly increasing on its domain and satisfies \( k(0) > \eta \).

**Proof.** First, note that \( k(0) = e^{-\eta} \), hence \( k(0) > \max\{\eta, b'(\eta)\} \geq \eta \). Second, \( k'(x) = e^{-\eta x} - b'(x) > e^{-\eta} - b'(\eta) > 0 \), because \( b \) is convex on the unit interval. \( \square \)

Equilibrium behavior is as follows. All \( \theta_1 \in \mathcal{D} \) reveal their identity via the invertible bribing function \( b \) and all higher types offer the common bribe \( b(\eta) \). Player 2 of type \( \theta_2 \) accepts a bribe \( B \) if and only if \( B \geq \theta_2 - k(b^{-1}(B)) \). Note that this implies that the maximal bribe which is offered in equilibrium, \( b(\eta) \), is accepted by all \( \theta_2 \); also, any other (strictly positive) equilibrium-bribe is accepted by some \( \theta_2 \)’s are rejected by others.\(^{36}\) When a bribe \( b(\theta_1) \) is rejected, player 1 bids \( k(\theta_1) \) in the auction that follows, and player 2 bids \( k(\theta_1)^+ \). This behavior can be completed, and beliefs be specified, so that a PBE obtains.

What happens in this equilibrium is that the types in the domain \( \mathcal{D} \) reveal their identity via their bribe, and then, in case that this bribe is rejected, they submit the bid \( k(\theta_1) \) in the auction. As Lemma 2 implies, the function \( k \) satisfies \( k(x) > x \) for all \( x \in \mathcal{D} \), hence dominated bidding strategies are employed. However, since a rejecting \( \theta_2 \) submits the bid \( k(\theta_1)^+ \), the dominated action brings no harm to player 1.

**Theorem 8.** Suppose that types are drawn independently from the uniform distribution on \([0, 1]\). Then, the first-price TIOLI game has a monotonic, continuous, bribery-involving equilibrium, in which dominated strategies are employed at the auction stage.

\(^{35}\)In the Appendix I prove that there exists a \( \eta > 0 \) such that (8) is satisfied.

\(^{36}\)Note that \( b(\eta) + k(\eta) = b(\eta) + (1 - b(\eta)) = 1 \), which implies that all \( \theta_2 \) accept this bribe. Next, note that \( \chi(t) \equiv b(t) + k(t) \), which is the threshold type that accepts \( b(t) \), is strictly increasing in \( t \); as we just saw above, \( \chi(\eta) = 1 \), which implies that any strictly positive bribe which is smaller than \( b(\eta) \) is accepted by some types of player 2 and rejected by other types.
The restriction to the uniform distribution is made for analytical convenience; one would expect analogous equilibrium-constructions, along the same lines, to be possible (however, more complicated) for other distributions too.

Under both the first-price and second-price formats the possibility of dominated (but optimal) behavior at the auction stage brings an improvement: in the former case it makes nontrivial equilibrium possible, and in the latter case it makes efficiency possible. On a more technical level, in the second-price game it gives rise to a fully revealing equilibrium, and in the first-price game it gives rise to a partially revealing equilibrium. Here, the italicized terms mean an equilibrium in which the bribing function is strictly increasing on its entire domain, and an equilibrium in which it is strictly increasing on a subset of its domain, respectively. One may wonder whether dominated bidding can “take us further” in the first-price game, to a fully revealing equilibrium. The answer is negative. It is formalized in the following result, which holds for any type distribution, not only uniform.

**Theorem 9.** The first-price TIO LI game does not have a fully revealing equilibrium, even if dominated bidding is allowed.

**Proof.** Assume by contradiction that such an equilibrium exists; let $b$ denote its bribing function. Let $T \equiv \{\theta_1 : b(\theta_1) > 1 - \theta_1\}$. By assumption, $T \neq \emptyset$; moreover, it is easy to see that $|T| > 1$. I argue that the bribe of each $\theta_1 \in T$ is accepted by all $\theta_2$. To see this, consider a $\theta_1 \in T$. Assume that $\beta \equiv b(\theta_1)$ is rejected in equilibrium by some $\theta_2$’s. Then, the post-rejection-of-$\beta$ is an on-path auction in which $\theta_1$ is common knowledge. It is easy to see that in such an auction player 1 bids at least $\theta_1$. Therefore, by rejecting $b(\theta_1)$ player 2 guarantees that his payoff will be at most $1 - \theta_1 < b(\theta_1)$, in contradiction to the fact that he is playing a best response. Therefore, $b(\theta_1)$ is accepted by all $\theta_2$’s, for all $\theta_1 \in T$. But this means that there can be at most one bribe of the form $b(\theta_1)$, $\theta_1 \in T$, which contradicts the fact that $b$ is strictly increasing and $|T| > 1$.

The logic behind this result is the one that prevents full revelation in the ES second-
price game, when only truthful bidding is allowed. The common thread that runs through both results is that full revelation is impossible in equilibrium if, following rejections, player 1 submits a bid which is at least as large as his valuation. In the second-price game he bids exactly his valuation when he employs his dominant strategy, and in the first-price game he may bid even higher. These bids can be lowered in the second-price game (by allowing for dominated bidding) but not in the first-price game.

4 Conclusion

I have studied a symmetric first-price auction which is preceded by a “take-it-or-leave-it” (TIOLI) bribing stage, an extensive form which is analogous to the one studied by Esö and Schummer (2004), where the auction format is second-price. The predictions of the two games differ along several dimensions.

When types are drawn from the unit interval, and under the restriction to pure and undominated strategies, and to continuous nondecreasing bribing functions, bribing can happen in equilibrium under the second-price format, but not under the first-price format. This is a consequence of a general principle, which applies to any protocol, not only TIOLI: in a second-price auction any information which was inferred from the pre-auction activity can be ignored if and when the auction is reached; in general, such information cannot be ignored in first-price auctions.

Under the aforementioned restrictions, information revaluation at the bribing stage creates the following incentives to cheat: the briber can mimic a smaller type, say \( x \), by offering the bribe \( b(x) \), which will likely be rejected by the respondent, who, consequently, will bid slightly above \( x \) in the auction. Then, the briber can bid slightly above the respondent, which enables him to obtain the good for a low price. Equilibrium existence is an issue under the first-price format, even in the absence of the aforementioned information-revelation problem: even a “no bribe equilibrium”
may fail to exist. Such an equilibrium can exist only if the types are, in a probabilistic sense, low. In such a case, the briber is not willing to pay “even a penny” for his rival’s abstention, and the no-bribing situation is sustained in equilibrium. In the second-price game, by contrast, a continuous bribery-involving equilibrium exists given any distribution that satisfies mild regularity conditions.\textsuperscript{37}

The existence problem can be remedied by the employment of dominated bidding strategies. I have shown that, at least under the uniform type distribution, this results in the existence of a bribery-involving, continuous, monotonic equilibrium. Another way to restore existence is by allowing mixed strategies; this possibility has not been explored in this paper. Finally, a third way to restore existence is by allowing for more pre-auction rounds.\textsuperscript{38} In Rachmilevitch (2010) I showed that when there are \emph{two} rounds of alternating bribe offers—i.e., when player 2 can offer a counterbribe if he rejects the first bribe—then, in a formal sense, it is irrelevant whether the auction format is first- or second-price. Under mild conditions on the type distribution, the two-round game has an equilibrium in which the auction is never reached; on its path, player 1 reveals his type through an invertible bribing function, player 2 employs an efficient acceptance rule, and his rejections are followed by counterbribes that are accepted by player 1. This behavior can be sustained in equilibrium under either auction format (provided the aforementioned distributional conditions).

When the minimal type is far from zero, bribing can be part of an equilibrium of the first-price TIOLI game. The general principle here is that under continuity and monotonicity only pooling equilibria are possible; since all types offer the same bribe in a pooling equilibrium, it is obvious that when the minimal type is zero then the corresponding common offer is zero, but if the minimal type is sufficiently large, then nontrivial bribing may occur.

\textsuperscript{37}The conditions of ES—that the type distribution be differentiable, strictly increasing, and log-concave—are sufficient for this result.

\textsuperscript{38}Formally speaking, of course, this is not restoring existence in our game, because by adding rounds one changes the game.
5 Appendix

Proof of Lemma 1: Let $b$ denote player 1’s bribing function in an equilibrium and let $J$ be a nondegenerate interval on which it is continuous. Assume by contradiction that $b$ is not constant on $J$. For obtaining a contradiction, the following preliminary steps are useful.

Step 1: If $\theta_1 \in J$ reveals himself through $b(\theta_1) \equiv \beta > 0$, then the set of $\beta$’s acceptors is an interval.

Proof of Step 1: If all the types of player 2 accept $\beta$, then clearly the set of $\beta$’s acceptors is an interval, $[0, 1]$. Suppose, on the other hand, that rejection of $\beta$ happens on the path. In this case, the post-rejection on-path auction is such that $\theta_1$ is common knowledge. Let $B$ denote $\theta_1$’s bid in the BNE of this auction. Let $\theta_2$ be an arbitrary type who accepts $\beta$ in equilibrium, and consider $\theta_2' < \theta_2$. Assume by contradiction that $\theta_2'$ rejects $\beta$ in equilibrium. This rejection implies $\theta_2' - B \geq \beta$. Therefore $\theta_2 - B > \beta$, in contradiction to the fact that $\theta_2$ accepts $\beta$ in equilibrium.

Therefore, if $\beta > 0$ is a revealing offer then the set of its acceptors is $[0, A]$, where $A = A(\beta)$. Note that $A(\beta) \geq \beta$.

Step 2: If $\theta_1 \in J$ reveals himself through $b(\theta_1) \equiv \beta > 0$ and his offer is rejected, and if such a rejection happens in equilibrium, then he bids his true valuation in the post-rejection auction.

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39 Such a type exists, because $\beta > 0$ and every $\theta_2 < \beta$ accepts it.
40 To see this, assume by contradiction that there exists such a $\beta$ for which $A(\beta) < \beta$. Consider a type $\theta_2 \in (A(\beta), \beta)$. Such a type is supposed to reject $\beta$ because it is strictly greater than $A(\beta)$, in contradiction to the fact that it is not incentive compatible to reject a bribe which exceeds ones valuation.
Proof of Step 2: Let \( \beta > 0 \) be a revealing offer which is rejected in equilibrium (by some types of player 2). Since the set of its acceptors is \([0, A(\beta)]\), it follows that in the post-rejection auction player 2’s type is distributed continuously on \([A(\beta), 1]\).

Let \( B \) denote player 1’s bid in this auction. Note that every \( \theta_2 \in [A(\beta), 1] \) submits the minimally winning bid in the auction, \( B^+ \) (because every such \( \theta_2 \) rejects a strictly positive bribe which means that his payoff in the auction is strictly positive, which means that he wins with a strictly positive probability). Since player 1 does not have a profitable deviation, any \( \epsilon \)-increase of \( B \) is non-profitable. Therefore, \( B = \theta_1 \). ■

Step 3: Let \( \theta_1 \) be in the interior of \( J \). If \( b(\theta_1) > 0 \) is a revealing offer, then there exist at least one type \( \theta_2 \) who rejects \( b(\theta_1) \) in equilibrium.

Proof of Step 3: Assume by contradiction that such a \( b(\theta_1) \) is accepted by all \( \theta_2 \in [0, 1] \). Since \( b \) is continuous, \( \theta_1 \) is in the interior of \( J \), and \( b \) is locally increasing at \( \theta_1 \), there exists a \( \theta'_1 \) in the interior of \( J \), \( \theta'_1 > \theta_1 \), who makes a revealing offer that satisfies \( b(\theta'_1) > b(\theta_1) \). Since this is a part of an equilibrium, it follows that the exists a \( \theta^*_2 \) who rejects \( b(\theta'_1) \) in equilibrium.\(^{41}\) As we saw in Step 2, player 1 bids his valuation truthfully in an on-path auction which follows revelation of his type. Therefore, the incentive constraints of the aforementioned \( \theta^*_2 \) imply \( b(\theta_1) \geq \theta^*_2 - \theta_1 > \theta^*_2 - \theta'_1 \geq b(\theta'_1) > b(\theta_1) \), a contradiction.\(^{42}\) ■

Therefore, it follows from Step 2 and Step 3, that when a revealing offer \( b(\theta_1) > 0 \) is rejected, where \( \theta_1 \) is in the interior of \( J \), player 1 bids his valuation truthfully in the post-rejection auction. Therefore, every \( \theta_2 \) for which \( \theta_2 - \theta_1 < b(\theta_1) \) accepts the bribe. That is, every \( \theta_2 < \theta_1 + b(\theta_1) \) accepts the bribe. Therefore, \( b^{-1}(\beta) + \beta \leq A(\beta) \) for every revealing \( \beta > 0 \). Moreover, I argue that the latter inequality is satisfied as

\(^{41}\)Otherwise, there is no reason to offer \( b(\theta'_1) \). Because, if both \( b(\theta'_1) \) and \( b(\theta_1) \) are accepted with probability 1, then only the lower bribe, \( b(\theta_1) \), can be offered in equilibrium.

\(^{42}\)The first weak inequality is due to the assumption that all types of player 2 accept \( b(\theta_1) \), and the second weak inequality is due to the fact that \( \theta^*_2 \) rejects \( b(\theta'_1) \).
equality. Otherwise, types $\theta_2 \in (b^{-1}(\beta) + \beta, A(\beta))$ accept the bribe even though it is strictly better for them to compete in the auction. Therefore, $A(\beta) = b^{-1}(\beta) + \beta$ for each revealing $\beta > 0$.

With the aforementioned preliminary steps established, we are now ready to prove that $b$ must be constant on $J$. If not, then by continuity and monotonicity there is a nondegenerate subinterval of $J$, $I \subset J$, on which $b$ is strictly increasing. For every $x \in I$ the (revealing) offer $b(x)$ is accepted by player 2 if and only if $\theta_2 \leq b(x) + x$. Therefore, when type $\theta_1 \in I$ makes a nondetectable deviation and mimics a nearby type $x \in I$, the expected payoff corresponding to this deviation is

$$\Psi(x|\theta_1) = \begin{cases} h(x|\theta_1) & \text{if } x \geq \theta_1 \\ h(x|\theta_1) + (1 - F(b(x) + x))(\theta_1 - x) & \text{if } x < \theta_1 \end{cases}$$

where $h(x|\theta_1) = F(b(x) + x)(\theta_1 - b(x))$. Since mimicking types $x > \theta_1$ is unprofitable, it follows that $\alpha \equiv \frac{d}{dx} h(x|\theta_1)|_{x=\theta_1} \leq 0$. Since mimicking types $x < \theta_1$ is unprofitable, it follows that $\alpha + \frac{d}{dx} \left[(1 - F(b(x) + x))(\theta_1 - x)\right]|_{x=\theta_1} \geq 0$, hence $\frac{d}{dx} \left[(1 - F(b(x) + x))(\theta_1 - x)\right]|_{x=\theta_1} \geq 0$. However, the latter expression is given by $- (1 - F(b(\theta_1) + \theta_1))$. Therefore, $F(b(\theta_1) + \theta_1) = 1$ at all differentiability points of $b$ in $I$.

This, however, is impossible. The reason is that there cannot be two types $x \in I$ of player 1 for whom $F(b(x) + x) = 1$. To see this, suppose that $\theta_1$ and $\theta'_1$ are such types with $\theta'_1 > \theta_1$. Since both are in $I$, $b(\theta'_1) > b(\theta_1)$. Since $F(b(x) + x) = 1$ for both $x \in \{\theta_1, \theta'_1\}$, both of these bribes are accepted with probability 1. But this means that type $\theta'_1$ has a profitable deviation: to offer $b(\theta_1)$. \hfill \Box

**Lemma 3.** Suppose that $F$ is differentiable and that its derivative, $f = F'$, is strictly

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Following the rejection of $b(x)$ all the rejecting types of player 2 submit the minimally winning bid, which in equilibrium coincides with the +-version of the revealed type of player 1, $x^+$. Thus, player 1 can guarantee a payoff arbitrarily close to $\Psi(x|\theta_1)$, by submitting the bid $x + \epsilon$ after $b(x)$ is rejected, where $\epsilon > 0$ is arbitrarily small. Thus, to be precise, $\Psi(x|\theta_1)$ is the supremum of the set of payoffs that player 1 can obtain. Alternatively, $\Psi(x|\theta_1)$ can be realized exactly, if we allow player 1 to submit bids of the form $r^{++}$, where the latter relates to $r^+$ in the same way that $r^+$ relates to $r$. 29
positive on \((0, 1)\). Then \(\mathbb{E}(\theta) \geq \frac{1}{2}\) if and only if \(\pi(\theta) < \frac{\theta}{2}\) for all \(\theta \in (0, 1)\).

**Proof.** Let \(F\) be such that \(F' = f > 0\) on \((0, 1)\). Suppose that \(\pi(\theta) < \frac{\theta}{2}\) \(\forall \theta \in (0, 1)\). Since \(\pi\) is continuous, \(\pi(1) \leq \frac{1}{2}\). Therefore \(1 - \mathbb{E}(\theta) \leq \frac{1}{2}\), or \(\mathbb{E}(\theta) \geq \frac{1}{2}\). Conversely, suppose that \(\mathbb{E}(\theta) \geq \frac{1}{2}\). We need to prove that \(\gamma(\theta) > 0\) for \(\theta \in (0, 1)\), where \(\gamma(\theta) \equiv \theta^2 - \pi(\theta)\). Note that \(\gamma'(\theta) = \frac{1}{2} - F(\theta)\), hence \(\gamma''(\theta) = -f(\theta) < 0\), so \(\gamma\) is strictly concave on \((0, 1)\). Also, \(\gamma(0) = 0\) and \(\gamma(1) = \frac{1}{2} - (1 - \mathbb{E}(\theta)) = \mathbb{E}(\theta) - \frac{1}{2} \geq 0\), where the inequality is by assumption. Therefore, \(\gamma(\theta) > \theta \gamma(1) + (1 - \theta) \gamma(0) = \theta \gamma(1) \geq 0\) for all \(\theta \in (0, 1)\). \(\square\)

**Remark:** For the general type space \([\underline{\theta}, \overline{\theta}]\) the double-implication analogous to the one from Lemma 3 is: \(\mathbb{E}(\theta) \geq \frac{\theta + \underline{\theta}}{2}\) if and only if \(\pi(\theta) < \frac{\theta - \underline{\theta}}{2}\) for all \(\theta \in (\underline{\theta}, \overline{\theta})\).

**Proof of Theorem 3:** Make the aforementioned assumptions on \(F\) and assume by contradiction that a restrictedly discontinuous equilibrium exists. Let \(b\) denote the equilibrium bribing function. Obviously, \(b(0) = 0\). Moreover, I argue that 0 is a continuity point of \(b\). To see this, assume by contradiction that it is not. Then, there is an \(\epsilon > 0\) and a sequence \(\{\theta^n\}_1\) such that \(\theta^n_1 \downarrow 0\) and \(b(\theta^n_1) \rightarrow \epsilon\). Pick an arbitrarily small element in the sequence, \(\theta^n_1 \equiv \delta\). For \(\delta\) sufficiently small we can assume that \(\delta < b(\delta)F(b(\delta))\). Note that type \(\delta\)’s expected payoff is bounded from above by \(F(b(\delta))(\delta - b(\delta)) + (1 - F(b(\delta)))\delta < 0\), in contradiction to equilibrium. Since 0 is a continuity point of \(b\) and the equilibrium is restrictedly continuous, \(b \equiv 0\) in a neighborhood of 0. I will now prove that all types of player 1 sufficiently close to zero have a profitable deviation to a positive bribe.

Let \(\sigma_2\) denote player 2’s strategy in this equilibrium. Consider a deviation by player 1 to a positive bribe \(b \in (0, 1]\) followed by the bid \(x\), in case that \(b\) is rejected;\footnote{In the symmetric (dominant strategy) equilibrium of the second-price auction each participant bids his valuation. Then, the fact that \(\pi(1) = 1 - \mathbb{E}(\theta)\) follows directly from the payoff equivalence of the first- and second-price formats.}
denote this deviation by $D = (b, x)$. The bribe $b$ is accepted by player 2 if and only if $\theta_2 \leq q$, for some threshold $q = q(b) \in [b, 1]$. Since the auction format is first-price, a type $\theta_2 > q$ who rejects $b$ does not bid more than $\theta_2 - b$ in the auction. Consider the alternative strategy for player 2, $\sigma_2$, which coincides with $\sigma_2$, except that it prescribes every $\theta_2 > q$ the bidding function $B(\theta_2) \equiv \theta_2 - b$. Clearly, player 1’s expected payoff from the deviation $D$, computed against $\sigma_2$, is at least as large as the one he would have obtained from this deviation if player 2’s strategy were $\sigma_2$. Therefore, any $D$ whose expected payoff, when computed against $\sigma_2$, is strictly greater than the equilibrium payoff $\pi(\theta_1)$, is a profitable deviation.

I will prove the existence of a type $\theta_1 \in (0, 1)$ for whom there exists a profitable-against-$\sigma_2$ deviation $D = (b, x)$. More specifically, I will prove that there exists such a deviation in the set of deviations that satisfy $x = b \leq \frac{\theta_1}{2}$; let $D(\theta_1)$ denote this set. When player 2 employs $\sigma_2$, $\theta_1$’s expected payoff from the deviation $D = (b, x) \in D(\theta_1)$ is

$$F(q)(\theta_1 - b) + (1 - F(q)) \times \left\{ \frac{\text{Prob}[x \geq B(\theta_2)|\theta_2 \geq q]}{P(D)} \right\}(\theta_1 - x).$$

(9)

Suppose that there exists a $\theta_1 \in (0, 1)$ for which there is a $D = (b, x) \in D(\theta_1)$ such that $q = q(b) = 1$. I argue that this $D$ is a profitable deviation for this $\theta_1$. To see this, assume by contradiction that it is not. Then, since in this case the payoff corresponding to the deviation is $\theta_1 - b$, it follows that

$$\theta_1 - b \leq \pi(\theta_1) < \frac{\theta_1}{2},$$

where the strict inequality follows from Lemma 3. Therefore $b > \frac{\theta_1}{2}$, in contradiction to $D \in D(\theta_1)$.

Suppose, then, that $q = q(b) < 1$ for every deviation $D = (b, x) \in D(\theta_1)$, for

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45 Obviously there is no reason to consider deviations to bribes $b > 1$.

46 By Lebrun (1999), the aforementioned assumptions on $F$ imply that there is a unique BNE in the symmetric first-price auction, which is the symmetric one; hence the equilibrium payoff is given by $\pi(.)$. 

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every $\theta_1 \in (0, 1)$. Look at $\text{Prob}[x \geq B(\theta_2)|\theta_2 \geq q] = \text{Prob}[\theta_2 \leq x + b|\theta_2 \geq q] = \max\{0, \frac{F(b+x) - F(q)}{1-F(q)}\} \equiv P(D)$. Since the deviations $D = (b, x) \in D(\theta_1)$ satisfy $b = x$, it follows that $P(D) > 0$ if and only if $2b > q(b)$, for every such $D$.

Case 1: There exist a sequence of positive numbers $\{b_n\}$, where $b_n \downarrow 0$, such that $2b_n > q(b_n)$. Let $D(\theta_1)^+ \equiv \{D = (b, x) \in D(\theta_1)|2b > q(b)\}$. Define the sequence of types of player 1, $\{\theta^n_1\}$, by $\theta^n_1 \equiv 2b_n$. Note that $(\theta^n_1, \frac{\theta^n_1}{2}) \in D(\theta_1)^+$ for every $\theta_1$ in that sequence. When such a type $\theta_1$ deviates to $D = (b, x) = (\frac{\theta_1}{2}, \frac{\theta_1}{2})$, the winning probability in the post-rejection-of-$b$ auction is $P(D) = \frac{F(b+x) - F(q)}{1-F(q)} > 0$, and the expected payoff (9) can be written as

$$F(q)(\theta_1 - b) + (F(b + x) - F(q))(\theta_1 - x) = F(q)(x - b) + F(b + x)(\theta_1 - x). \quad (10)$$

Since $x = b = \frac{\theta_1}{2}$, the RHS of (10) equals $F(\theta_1)\frac{\theta_1}{2}$.

I will prove that $F(\theta_1)\frac{\theta_1}{2} > \pi(\theta_1) = \int_0^{\theta_1} F(t)dt$ for all sufficiently small $\theta_1 > 0$. Since $F(0)\frac{\theta_1}{2} = \pi(0) = 0$, it is enough to show that there exists an $\epsilon > 0$, such that $\frac{d}{d\theta_1}[F(\theta_1)\frac{\theta_1}{2}] \geq \pi'(\theta_1) = F(\theta_1)$ on $[0, \epsilon]$, with a strict inequality on $(0, \epsilon) \equiv I$. That is, that $\nu(\theta_1) \equiv f(\theta_1)\theta_1 > F(\theta_1)$ for $\theta_1 \in I$. Since $\nu(0) = 0 = F(0)$, it is enough to prove that $\nu' > F' = f$ on $I$. Namely, that $f'(\theta_1)\theta_1 > 0$ for $\theta_1 \in I$. This follows from $f'(0) > 0$.

Case 2: There exist a sequence of positive numbers $\{b_n\}$, where $b_n \downarrow 0$, such that $2b_n \leq q(b_n)$ for every $n$. Let $D(\theta_1)^0 \equiv \{D = (b, x) \in D(\theta_1)|2b \leq q(b)\}$. Define the sequence of types of player 1, $\{\theta^n_1\}$, by $\theta^n_1 \equiv 2b_n$. Note that $(\frac{\theta_1}{2}, \frac{\theta_1}{2}) \in D(\theta_1)^0$ for every $\theta_1$ in that sequence. When such a type $\theta_1$ deviates to $D = (b, x) = (\frac{\theta_1}{2}, \frac{\theta_1}{2})$, the winning probability in the post-rejection-of-$b$ auction is $P(D) = 0$, and the expected payoff (9) is $F(q)(\theta_1 - b) \geq F(2b)(\theta_1 - b) = F(\theta_1)\frac{\theta_1}{2}$. The arguments from Case 1 complete the proof. □
Proof of Proposition 1: Let $\tilde{b}$ be the following bribing function: $\tilde{b}(x) = \mathbb{E}(\theta|\theta \leq x)$ for $x < 1$ and $\tilde{b}(1) = 0$. Consider the following strategy. Player 1 offers bribes according to $\tilde{b}$, and bids in the auction that follows the rejection of the offer $B$ as follows: if $B = \tilde{b}(\theta_1)$ (namely, if he did not deviate at the first stage), then he bids $\theta_1 - \tilde{b}(\theta_1)$ in the auction; if, on the other hand, $B \neq \tilde{b}(\theta_1)$, then he bids truthfully in the auction. When player 2 sees the offer $B \in (0, \mathbb{E}(\theta))$, he accepts it if and only if $\tilde{b}^{-1}(B) > \theta_2$; the offer $B = 0$ is rejected and $B \geq \mathbb{E}(\theta)$ is accepted. Player 2 bids truthfully in the auction, unless $B = 0$, in which case he bids $\mathbb{E}(\theta)$.

Let player 2 believe that $\theta_1 = \tilde{b}^{-1}(B)$ for $B \in (0, \mathbb{E}(\theta))$, let him adopt any arbitrary belief on the support $\{0, 1\}$ if $B = 0$ and let him believe that $\theta_1 = 1$ if $B \geq \mathbb{E}(\theta)$. In case that $B \in (0, \mathbb{E}(\theta))$ was rejected, let player 1 adopt the belief that $\theta_2$ is distributed on $[\tilde{b}^{-1}(B), 1]$ according to the conditional distribution $F_{\{\theta_2 \geq \tilde{b}^{-1}(B)\}}$; in case that $B = 0$ was rejected let his belief be given by the prior distribution $F$, and in case that $B \geq \mathbb{E}(\theta)$ was rejected let him assign probability 1 to $\{\theta_2 = 1\}$.

I argue that the aforementioned description amount to a PBE. It is clear that beliefs obey Bayes’ rule, hence we only need to verify mutual best-responses. Regarding bidding, there are only two cases in which a player is instructed to bid nontruthfully in the auction. The first case is where player 1 is instructed to bid less than his valuation in an auction in which he expects player 2 to submit a bid which is greater than $\theta_1$ with certainty (this only happens in auctions prior to which player 1 adhered to the equilibrium bribing function and revealed his type to player 2, and player 2 rejected the bribe, which happens on the path only when $\theta_2 > \theta_1$). The second such case is where player 2 is instructed to bid $\mathbb{E}(\theta)$ in the auction that follows the rejection of $B = 0$. Recall that in this auction his belief is given by some distribution on the support $\{0, 1\}$. Note that type $\theta_1 = 0$ submits the bid zero, against which the bid $\mathbb{E}(\theta)$ is optimal for player 2. Also, type $\theta_1 = 1$ bids truthfully, against which the bid $\mathbb{E}(\theta)$ is optimal for player 2. Therefore, the bidding of each player in every continuation auction (on and off the path) is optimal for him. It is left to verify that
the acceptance rule of player 2 is optimal for him and that the bribing function $\tilde{b}$ is ex ante optimal for player 1.

Ex-ante optimality for player 1: Consider type $\theta_1 \in (0, 1)$. Suppose that he mimics type $x \in [\theta_1, 1)$. His expected payoff is $F(x)\theta_1 - \int_0^x tf(t)ft$. The first order condition is $f(x)\theta_1 - f(x)x = 0$, hence the maximum on this range is at $x = \theta_1$.\(^{47}\) If he mimics a type $x \in (0, \theta_1]$ the corresponding payoff is $F(x)\theta_1 - \int_0^x tf(t)ft + (F(\theta_1) - F(x))(\theta_1 - \mathbb{E}(\theta_2|\theta_2 \in (x, \theta_1)))$, or $F(\theta_1)\theta_1 - \int_0^{\theta_1} tf(t)dt = \pi(\theta_1)$. Mimicking the types $\theta_1 \in \{0, 1\}$ also give the expected noncooperative payoff $\pi(\theta_1)$, which coincides with his equilibrium payoff. Therefore, if there is a profitable deviation at the bribing stage, then it must be a publicly detectable deviation; namely a deviation to a bribe $B \geq \mathbb{E}(\theta)$. This deviation is accepted with probability 1 and gives the payoff $\theta_1 - B \leq \theta_1 - \mathbb{E}(\theta)$. Therefore, it suffices to show that $\theta_1 - \mathbb{E}(\theta) \leq \pi(\theta_1)$. Note that this is inequality can be written as $\theta_1 - \int_0^1 tf(t)dt \leq \int_0^{\theta_1} (\theta_1 - t)f(t)dt$, or, upon rearranging, as $[1 - F(\theta_1)]\theta_1 \leq \int_0^{\theta_1} tf(t)dt$.

Both sides equal zero at $\theta_1 = 1$, therefore it is enough to prove that the derivative of the LHS is greater than that of the RHS. Indeed, $-f(\theta_1)\theta_1 + [1 - F(\theta_1)] \geq -f(\theta_1)\theta_1$.

Optimality of player 2’s acceptance rule: Consider first an offer $B \in (0, \mathbb{E}(\theta))$. Acceptance gives the payoff $B$. The expected payoff in the auction that follows its rejection is $\max\{0, \theta_2 - (\hat{\theta}_1 - \tilde{b}(\hat{\theta}_1))\}$, where $\hat{\theta}_1 = \tilde{b}^{-1}(B)$. It is straightforward that accepting the offer if $\hat{\theta}_1 > \theta_2$ and rejecting it otherwise is optimal. Acceptance of $B \geq \mathbb{E}(\theta_1)$ is supported as optimal by the belief that player 1 is of the maximal type, and rejection of $B = 0$ is clearly optimal. \(\square\)

The rest of the Appendix is devoted to proving Theorem 8. To this end, the following preliminary results will be helpful.

**Lemma 4.** The function $b$, defined by (6), is strictly increasing.

**Proof.** Note that $b'(0) = 0$ and $b''(0) = 1 - b'(0) = 1$, hence $b$ is locally increasing.

\(^{47}\)The second-order condition is obviously satisfied.
near zero. Then, if there is a region where \( b \) is nonincreasing, then there is a point \( x > 0 \) which is a local maximum; this maximum may be either unique or not unique. In the first case, \( b'(x) < 0 \) and in the latter case \( b''(x) = 0 \); however, neither one is possible, because \( b''(x) = 1 - b'(x) = 1 \).

Note that Lemma 4 implies \( b(t) < t \), which is to be expected: equilibrium-bribes always fall short of the briber’s valuation.

**Lemma 5.** The function \( b \), defined by (6), is convex on the unit interval.

**Proof.** Assume by contradiction that \( b''(x) = 1 - b'(x) \leq 0 \) for some \( 0 < x \leq 1 \). Then, \( 1 \leq b'(x) = x - b(x) < x \leq 1 \), a contradiction.

**Lemma 6.** There exists an \( \eta > 0 \) such that (8) is satisfied.

**Proof.** Look at \( e^{-\eta+x}b(x) + (1 - e^{-\eta+x})(e^{-\eta+x} - b(x)) \geq b(\eta) \). First, note that the LHS is decreasing in \( x \). To see this, note that its derivative with respect to \( x \) is

\[
e^{-\eta+x}b(x) + e^{-\eta+x}b'(x) - e^{-\eta+x}(e^{-\eta+x} - b(x)) + (1 - e^{-\eta+x})(e^{-\eta+x} - b'(x))
\]

Evaluating this expression at \( \eta = x = 0 \) gives \(-1\). Hence, for a sufficiently small \( \eta > 0 \) this expression is indeed negative, uniformly over \( x \in [0, \eta] \). Therefore, the LHS of the aforementioned inequality is minimized at \( x = \eta \), where it assumes the value \( b(\eta) \).

Given \( \theta_1 \in D \), define the function \( f_{\theta_1}: D \to \mathbb{R} \) by \( f_{\theta_1}(x) \equiv e^{-\eta+x}(\theta_1 - b(x)) \).

**Lemma 7.** For all \( \theta_1 \in D \), \( f_{\theta_1}(x) \) is maximized at \( x = \theta_1 \).

**Proof.** \( f'_{\theta_1}(x) = e^{-\eta+x}(\theta_1 - b(x) - b'(x)) \), and therefore \( f_{\theta_1}(x) = 0 \) if and only if \( \theta_1 - b(x) = b'(x) \). By definition of \( b \), this equation is satisfied at \( x = \theta_1 \); also, by Lemma 4 the LHS is decreasing in \( x \) and by Lemma 5 the RHS is increasing in \( x \). Therefore, there is a unique point where \( f_{\theta_1}(x) = 0 \): \( x = \theta_1 \). Hence, it is enough to prove that this point is local maximum, or \( f''_{\theta_1}(\theta_1) < 0 \). Note that

\[
f''_{\theta_1}(x) = e^{-\eta+x}(\theta_1 - b(x) - b'(x)) + e^{-\eta+x}[-b'(x) - b''(x)] = e^{-\eta+x}(\theta_1 - b(x) - b'(x)) + e^{-\eta+x}[-b'(x) - (1 - b'(x))],
\]

hence \( f''_{\theta_1}(\theta_1) = -e^{-\eta+\theta_1} < 0 \).
We are now ready to turn to the theorem’s proof. In it, I introduce one more special bid, \( r^{++} \); it relates to \( r^+ \) in the same way that the latter relates to \( r \).

**Proof of Theorem 8**: Recall the strategy from the text: player 1 is instructed to offer the bribe \( b(\min\{\theta_1, \eta\}) \) and is instructed to bid \( k(\min\{\theta_1, \eta\}) \) if the bribe is rejected. Player 2 accepts a bribe \( B \leq b(\eta) \) if and only if \( B \geq \theta_2 - k(b^{-1}(B)) \) and bribes greater than \( b(\eta) \) are accepted by all \( \theta_2 \). In the auction that follows a rejection of the bribe \( B \), behavior is as follows. For \( B \leq b(\eta) \), player 2 bids \( k(b^{-1}(B))^+ \) and player 1 bids \( k(b^{-1}(B)) \) if it is a best-response against \( k(b^{-1}(B))^+ \) (i.e., if \( \theta_1 \leq k(b^{-1}(B)) \)) and bids \( k(b^{-1}(B))^++ \) otherwise. For \( B > b(\eta) \) the players’ beliefs are unrestricted, so any specification of bids and beliefs supporting these beliefs will do.

Consider player 1 when he makes the first move in the game. Suppose first that \( \theta_1 \in D \). Clearly there is no reason to offer more than \( b(\eta) \), so the only question is which type in \( D \) it is optimal to mimic; by Lemma 7, it is optimal to “mimic” oneself.\(^{48}\) Consider now \( \theta_1 \notin D \). Consider the deviation that mimics type \( x < \eta \). The payoff associated to this deviation is \( e^{-\eta+x}(\theta_1 - b(x)) + (1 - e^{-\eta+x})\max\{0, \theta_1 - k(x)\} \). Let us divide all possible deviations \( x \in [0, \eta] \) into two classes: those such that \( k(x) > \theta_1 \) and those for which \( k(x) \leq \theta_1 \).\(^ {49} \) Look at a deviation of the former kind. It follows from the proof of Lemma 7 that the derivative of the payoff function is strictly increasing at any such \( x \), and hence its associated payoff cannot be greater than the equilibrium payoff, the one that corresponds to \( x = \eta \). Consider, on the other hand, a deviation of the latter kind. The payoff corresponding to this deviation is \( \theta_1 - e^{-\eta+x}b(x) - (1 - e^{-\eta+x})(e^{-\eta+x} - b(x)) \). By (8), this expression is smaller than the equilibrium payoff, \( \theta_1 - b(\eta) \). Hence, player 1 is playing optimally when making the first move in the game. It is clear that the players play best-responses in all other information sets. \( \square \)

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\(^{48}\)Recall Lemma 2: \( k \) is increasing and satisfies \( k(0) > \eta \). Therefore \( f_{\theta_1} \) is \( \theta_1 \)'s payoff function—because following the rejection of \( b(x) \) player 2 bids (the +-version of) \( k(x) \geq k(0) > \eta \geq \theta_1 \).

\(^{49}\)It may be the case that one of these sets (but not both) is empty.
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6 References


