A Behavioral Arrow Theorem

Alan D. Miller* and Shiran Rachmilevitch‡

May 23, 2012

Abstract

In light of research indicating that individual behavior may violate standard assumptions of rationality, we modify the standard model of preference aggregation to study the case in which neither individual nor collective preferences are required to satisfy transitivity or other coherence conditions. We introduce the concept of an ordinal rationality measure which can be used to compare preference relations in terms of their level of coherence. Using this measure, we introduce a monotonicity axiom which requires that the collective preference become more rational when the individual preferences become more rational. We show that for any ordinal rationality measure, it is impossible to find a collective choice rule which satisfies the monotonicity axiom and the other standard assumptions introduced by Arrow (1963): unrestricted domain, weak Pareto, independence of irrelevant alternatives, and nondictatorship.

Keywords: Aggregation; Axioms; Intransitivity; Coherence; Monotonicity; Rationality; Arrow's Theorem.

JEL Codes: D60; D70; D71.

*Faculty of Law and Department of Economics, University of Haifa, Mount Carmel, Haifa, 31905, Israel. Email: admiller@econ.haifa.ac.il Web: http://econ.haifa.ac.il/∼admiller/

†Department of Economics, University of Haifa, Mount Carmel, Haifa, 31905, Israel. Email: shiranrach@econ.haifa.ac.il Web: http://econ.haifa.ac.il/∼shiranrach/

‡Acknowledgements will come in a final version.

1 Introduction

The core model of preference aggregation introduced by Arrow (1963) contains a strong assumption about the rationality of preferences. In particular, individual preferences are assumed to be reflexive, complete, and transitive. These first two properties are often considered richness conditions, while transitivity is a coherence condition. (See Bossert and Suzumura (2007).) In light of behavioral research casting doubt on the assumption of transitivity (Tversky, 1969), we modify Arrow's model to remove the requirement that individual preference relations be transitive or satisfy other known coherence conditions.

Having removed this rationality requirement, we use this new framework to study an important question about the coherence of collective preferences. Do more rational individuals create more rational society? We illustrate this question by means of a simple example.

It is clear that many collective choice rules satisfy the remaining assumptions imposed by Arrow (1963): unrestricted domain, weak Pareto, independence of irrelevant alternatives, and nondictatorship. A simple example is the method of majority decision, in which alternative x is weakly preferred to alternative y whenever the majority weakly prefers x to y. (For more on the method of majority decision, see Sen (1964, 1966).)

However, the method of majority decision has an undesirable property. Suppose that there are three individuals, Alice, who prefers x to y to z, Bob, who prefers z to x to y, and Carol who prefers x to z to y to x. Alice and Bob have transitive preferences, but Carol's preferences are not. By the method of majority decision, x is preferred to y, z is preferred to y, and x is preferred to z, leading to a transitive and rational collective preference. However, suppose that Carol realizes that her preferences are irrational and seeks to "correct" them. She decides to retain her view that y is preferred to x but changes her opinion of z, so that she now prefers y to z and z to x. As a consequence, the method of majority decision leads to the collective preference x to y to z to x, and is no longer transitive. In this case, the collective preference became less rational *because* Carol became more rational.

We show that this problem is not unique to method of majority decision. In fact, every collective choice rule which satisfies the remaining assumptions of Arrow (1963) will have this undesirable property. For every such method it is possible that an increase in the coherence of individual preferences will lead to an decrease in the coherence of the collective preference.[1]

Our formal model can be described as follows. First, we study a modified version

[1]For this reason, we do not argue that the method of majority decision is any worse that any other method in this context. Several studies, including Sen (1966), Inada (1969), and Batra and Pattanaik (1972), examine the conditions under which pairwise majority does not lead to cycles. Dasgupta and Maskin (2008) provide an argument that the method of majority decision is more robust than other voting methods in that it violates the standard axiom on fewer domains.

of collective choice rules in which neither the individual nor collective preferences are required to be rational. We assume only that preference relations be reflexive and complete. Thus each individual's preferences can be described by a reflexive and complete relation, and the the collective preference can be described by a reflexive and complete relation as well. We implicitly assume that every possible combination of individual preference relations is possible; i.e. that Arrow's unrestricted domain axiom holds in this setting.

Using accepted notions of rationality, we introduce the concept of an ordinal rationality measure, and identify some minimal conditions that any reasonable rationality measure should satisfy. We formulate an axiom, *monotonicity*, to address the problem exhibited by majority rule in the above example. This axiom requires that the collective choice rule be monotonic with respect the rationality measure; that is, if individual preferences change and become more rational, then the collective preference should become more rational, if it changes at all.

In addition to monotonicity, we impose the three additional axioms of Arrow (1963): *weak Pareto*, which requires that society strictly prefer x to y whenever every individual strictly prefers x to y, *independence of irrelevant alternatives*, which requires that a change in the opinions about alternative z does not affect the relative ranking of alternatives x and y, and *nondictatorship*, which requires that no individual be a dictator. We show that the four axioms are incompatible. In other words, regardless of which ordinal rationality measure we choose, we cannot find a collective choice rule which is monotonic, weakly Paretian, independent of irrelevant alternatives, and nondictatorial.

1.1 Relevant Literature

Previous studies have sought to weaken the assumption of rationality in Arrow (1963) by permitting a wider range of collective preferences. The case of quasi-transitive collective preferences was studied by many including Gibbard (1969), Sen (1969, 1970), Schick (1969), and Mas-Colell and Sonnenschein (1972), and of acyclic preferences by Mas-Colell and Sonnenschein (1972) and Blair and Pollak (1982). For more see Sen (1977).

Other scholars have tried to avoid the negative conclusions of Arrow (1963) by moving in the opposite direction. Instead of expanding the range of admissible collective preferences, these studies restrict the domain of allowable preferences. The most prominent example is that of the single-peaked preference restriction of Black (1948a,b) and Arrow (1963).

The most closely related literature is the study of tournaments, which are described by binary relations which are antisymmetric and complete. Unlike the preference relations we study, tournaments do now allow for the possibility of ties. In this context, Monjardet (1978) shows that a collective choice rule that (a) maps every profile of transitive preferences into a transitive preference, (b) satisfies the independence of irrelevant alternatives axiom and (c) satisfies a non-imposition axiom is either dictatorial or "persecutive." Roughly speaking, persecutive means that the decisive coalitions are all the coalitions that do not contain a certain individual i. A related result can be found in Barthelemy (1982). As far as we can tell, the monotonicity axiom that we present is new to this paper.

The violation of transitivity is one of the simplest and most basic violations of rationality. There are, of course, more sophisticated violations. Rubinstein and Salant (forthcoming), for example, consider a decision maker whose behavior is consistent with the maximization of one (transitive) preference relation under some circumstances, and consistent with the maximization of another (transitive) preference relation under other circumstances. Related works by Manzini and Mariotti (2007), Cherepanov et al. (2008), and Masatlioglu et al. (forthcoming) consider a decision maker who first identifies a subset of alternatives from the grand set of all alternatives, and then maximizes a transitive relation on this subset.

2 Model and result

Let X be a set of alternatives, $|X| \geq 3$. A binary relation R on X is (a) *complete* if for all $x, y \in X$, $x \neq y$ implies that either xRy or yRx, (b) *reflexive* if for all $x \in X$, xRx, and (c) *transitive* if for all $x, y, z \in X$, xRy and yRz implies that xRz. Let $\mathcal{R}$ be the set of all complete and reflexive binary relations on X. Let $\mathcal{R}^*$ be the set of all complete and transitive binary relations on X.[2] For simplicity, we will refer to elements of $\mathcal{R}$ as *preference relations* and to elements of $\mathcal{R}^*$ as *preference orderings*. Clearly, every preference ordering is a preference relation; that is, $\mathcal{R}^* \subseteq \mathcal{R}$. For a preference relation $R \in \mathcal{R}$ we denote by P its asymmetric component; that is, xPy if xRy but not yRx.

For $Y \subseteq X$, denote by $\mathcal{R}|_Y$ the set all preference relations on Y, and denote by $\mathcal{R}^*|_Y$ the set all preference orderings on Y. For $R \in \mathcal{R}$ and $Y \subseteq X$, denote by $R|_Y \in \mathcal{R}|_Y$ the restriction of R to Y.

Let $N \equiv \{1, ..., n\}$ be a finite set of agents, $n \geq 2$. A *profile* $R = (R_1, \cdots, R_n) \in \mathcal{R}^N$ is a vector of binary relations, one for each agent. An *collective choice rule* is a mapping $f\colon \mathcal{R}^N \to \mathcal{R}$.[3] We define $R_0 \equiv f(R)$ to be the social relation, and we write P_0 to denote its asymmetric component.

A **rationality measure** is a binary relation $\succeq$ on $\mathcal{R}$ which satisfies the following properties:

1. For all $R \in \mathcal{R}$, $R \succeq R$.

[2] The results in this paper would not change if we replaced 'transitive' with 'quasi-transitive' or 'acyclic'. For more see subsection 2.1, below.

[3] This is a slight abuse of notation. Typically, the domain of a collective choice rule is a set of preference orderings. See Sen (1970).

2. For all $R^* \in \mathcal{R}^*$ and $R \in \mathcal{R}$, $R \succeq R^*$ implies that $R \in \mathcal{R}^*$.

3. For all $R^* \in \mathcal{R}^*$ and $R \in \mathcal{R} \backslash \mathcal{R}^*$, $R^* \succeq R$.

For two profiles $R, R' \in \mathcal{R}^N$ we write $R \succeq R'$ if $R_i \succeq R'_i$ for all $i \in N$.

Property 1, known as reflexivity, requires each preference relation to be "at least as rational" as itself. Property 2 requires that only a preference ordering can be at least as rational as another preference ordering. Property 3 requires that every preference ordering be at least as rational as every non-transitive preference relation.[4]

A wide range of rationality measures satisfies these conditions. The simplest rationality measure $\succeq'$ is one for which $R^* \succeq' R$ if and only if $R^* \in \mathcal{R}^*$ and $R \in \mathcal{R} \backslash \mathcal{R}^*$. A more complicated rationality measure can incorporate the structure of coherence properties studied in the social choice literature. Define $\mathcal{R}^q$ and $\mathcal{R}^a$ as the sets of quasi-transitive and acyclic preference relations, respectively.[5] Then $\mathcal{R} \subsetneq \mathcal{R}^q \subsetneq \mathcal{R}^a \subsetneq \mathcal{R}$. (See Suzumura (1983).) Thus, we can define a rationality measure $\succeq''$ such that $R^* \succeq'' R$ if and only if there exists an $\mathcal{C} \in \{\mathcal{R}^*, \mathcal{R}^q, \mathcal{R}^a\}$ such that $R^* \in \mathcal{C}$ but $R \notin \mathcal{C}$.[6]

Our first axiom, monotonicity, requires that if preference relations change, and each individual's new preference relation stays at least as rational as it was before the change, then the social preference must stay at least as rational.

Monotonicity: For all $R, R' \in \mathcal{R}^N$, if $R \succeq R'$ then $R_0 \succeq R'_0$.

The following three axioms were introduced by Arrow (1963); for brevity, we will not discuss them.

Weak Pareto: For every $R \in \mathcal{R}^N$ and $x, y \in X$, if xP_iy for all $i \in N$, then xP_0y.

Independence of Irrelevant Alternatives: For all $Y \subseteq X$ and $R, R' \in \mathcal{R}^N$, if $R|_Y = R'|_Y$, then $R_0|_Y = R'_0|_Y$

An individual $d \in N$ is a *dictator* if, for all $R \in \mathcal{R}^N$, xP_dy implies that xP_0y.

Non-Dictatorship: There does not exist a dictator.

We can now turn to the main result.

[4]Properties 2 and 3 can be weakened without changing our results. This is discussed Section 2.1, below.

[5]A binary relation is quasi-transitive if, for all $x, y, z \in X$, xPy and yPz together imply that xPz. A binary relation is acyclic if for every $k \geq 3$ and every $x^1, ..., x^k \in X$, x^iPx^{i+1} for all $i < k$ implies that x^kPx^1 does not hold.

[6]The four classes were chosen for the ease of the exposition, clearly a rationality measure can incorporate any number of classes, and these not be totally ordered through set inclusion. In particular, the rationality measure can incorporate the coherence properties of semi-transitivity and the interval order. See Cato (2011).

Theorem 1. *There does not exist a collective choice rule f that satisfies monotonicity, weak Pareto, independence of irrelevant alternatives, and non-dictatorship.*

To prove this Theorem we make use of the following lemma. For a coalition $K \subseteq N$, we define $x\bar{D}_K y$ as the statement that the coalition K is decisive for x over y; that is, if $xP_i y$ for all $i \in K$, then $xP_0 y$. Similarly, we define $xD_K y$ as the statement that the coalition K is decisive for x over y when all others are opposed; that is, if $xP_i y$ for all $i \in K$ and $yP_i x$ for all $i \notin K$, then $xP_0 y$.

Lemma 1. *If a collective choice rule f satisfies monotonicity, weak Pareto, and independence of irrelevant alternatives, then whenever $xD_K y$ for a coalition $K \subseteq N$ and some pair of alternatives $x, y \in X$, it follows that $w\bar{D}_K z$ for every pair $w, z \in X$.*

The proofs of the theorem and the lemma are given in the appendix.

2.1 Weakening the axioms.

The monotonicity axiom can be weakened to allow for a broader class of rationality measures. In particular, properties 2 and 3 can be weakened to properties $2'$ and $3'$. Recall that $\mathcal{R}^a$ is the set of all acyclic preference relations; that is, those which do not contain P-cycles. A set of elements $Y \subseteq X$ is *top-ranked* in profile R if $a \in Y$, $b \in X \setminus Y$, and $i \in N$ implies that $aP_i b$.

- $2'$. For all $R^* \in \mathcal{R}^*$ and $R \in \mathcal{R}$, $R \succeq R^*$ implies that $R \in \mathcal{R}^a$.
- $3'$. For all $R^* \in \mathcal{R}^*$ and $R \in \mathcal{R} \backslash \mathcal{R}^a$: if there is a three-element set $Y \subset X$ which is top-ranked in both relations, such that $R^* \big|_{X \backslash Y} = R \big|_{X \backslash Y}$, then $R^* \succeq R$.

Property $2'$ requires that only an acyclic preference relation can be at least as rational as a transitive preference ordering. Because every transitive preference relation is also acyclic, this property is weaker than property 2. Property $3'$ changes property 3 in two ways. First, it applies only to comparisons between transitive preference orderings and cyclic preference relations. Second, it is limited to the specific case in which the cyclic preference relation and the transitive preference ordering are identical except for the three top-ranked elements. This is a very clear case in which the transitive relation is more rational than the cyclic one.

By weakening the properties of the rationality measure, we consequently weaken the monotonicity axiom. This would not, however, change the main result. There does not exist a collective choice rule f that satisfies monotonicity (with respect to any rationality measure satisfying 1, $2'$ and $3'$), weak Pareto, independence of irrelevant alternatives, and non-dictatorship.

3 Conclusion

This paper departs from the standard approach to preference aggregation in three ways. First, in light of research indicating that individual behavior may violate standard assumptions of rationality, we modify the standard model of preference aggregation to study the case in which neither individual nor collective preferences are required to satisfy transitivity or other coherence conditions. Second, we introduce the concept of an ordinal rationality measure which can be used to compare preference relations in terms of their level of coherence. Third, using this measure, we introduce a monotonicity axiom which requires that the collective preference become more rational when the individual preferences become more rational. We show that for any ordinal rationality measure, it is impossible to find a collective choice rule which satisfies the monotonicity axiom and the other standard assumptions introduced by Arrow (1963): unrestricted domain, weak Pareto, independence of irrelevant alternatives, and nondictatorship.

This conclusion has practical implications. For example, one may wonder whether a group of people becomes less susceptible to "Dutch books" when the individuals' susceptibility lessens. Our result indicates that if the group decisions are made in a non-dictatorial way, it is possible that an increase in individual rationality may lead to a decrease in collective rationality. It may be possible to manipulate a group by helping individuals correct their mistakes.

Appendix

Proof of Lemma 1. We assume that the rationality measure satisfies properties 1, $2'$, and $3'$. This will be sufficient to prove the lemma.

Let the collective choice rule f satisfy the monotonicity, weak Pareto, and independence of irrelevant alternatives axioms. Let $K \subseteq N$ and $x, y \in X$ such that $xD_K y$.

Step one. We claim that, for all $z \in X \setminus \{x, y\}$, if $R \in \mathcal{R}^N$ such that (a) $R_i\big|_{\{x,y,z\}} \in \mathcal{R}^*\big|_{\{x,y,z\}}$ for all $i \in N$, (b) $xP_i y$ for all $i \in K$, and (c) $R_i\big|_{\{x,y,z\}} = R_j\big|_{\{x,y,z\}}$ for all $i, j \in K$, then $R_0\big|_{\{x,y,z\}} \in \mathcal{R}^*\big|_{\{x,y,z\}}$. To prove this claim, let $z \in X \setminus \{x, y\}$ and let $R \in \mathcal{R}^N$ satisfying (a), (b), and (c). From the independence of irrelevant alternatives axiom we can assume, without loss of generality, that the set $\{x, y, z\}$ is top-ranked in each R_i. Let $R^\circ \in \mathcal{R}^N$ such that (i) $R_i^\circ = R_i$ for all $i \in K$, (ii) $yP_i^\circ x$, $xP_i^\circ z$, and $zP_i^\circ y$ for all $i \notin K$, and (iii) $R_i \succeq R_i^\circ$ for all $i \in N$. Because $xD_K y$ it follows that $xP_0^\circ y$.

From condition (c) it follows that there are two cases: either $xP_i^\circ z$ for all $i \in K$, or $zR_i^\circ x$ for all $i \in K$. In the former case, $xP_i^\circ z$ for all $i \in N$, which implies (by weak Pareto), that $xP_0^\circ z$. Because $xP_0^\circ y$ and $xP_0^\circ z$, it follows that $R_0^\circ\big|_{\{x,y,z\}} \in \mathcal{R}^*\big|_{\{x,y,z\}}$. In the latter case, $zP_i^\circ y$ for all $i \in N$, which implies (by weak Pareto), that $zP_0^\circ y$. Because $xP_0^\circ y$ and $zP_0^\circ y$, it follows that $R_0^\circ\big|_{\{x,y,z\}} \in \mathcal{R}^*\big|_{\{x,y,z\}}$. Because $R_0^\circ\big|_{\{x,y,z\}} \in$

$\mathcal{R}^*\big|_{\{x,y,z\}}$ it follows from monotonicity and independence of irrelevant alternatives that $R_0\big|_{\{x,y,z\}} \in \mathcal{R}^*\big|_{\{x,y,z\}}$, proving the claim.

Step two. Let $R' \in \mathcal{R}^{*N}$ such that, for all $i \in K$, $xP_i'y$ and $yP_i'z$ and, for all $i \notin K$, $yP_i'x$ and $yP_i'z$. Because xD_ky it follows that $xP_0'y$, and because $yP_i'z$ for all $i \in N$ it follows from weak Pareto that $yP_0'z$. Because R' satisfies requirements (a), (b), and (c) of step one, it follows that $R_0'\big|_{\{x,y,z\}} \in \mathcal{R}^*\big|_{\{x,y,z\}}$ and therefore $xP_0'z$. By the independence of irrelevant alternatives axiom, this implies that $x\bar{D}_Kz$. In other words:

$$xD_Ky \text{ implies that } x\bar{D}_Kz. \tag{1}$$

Now, let $R'' \in \mathcal{R}^{*N}$ such that, for all $i \in K$, $zP_i''x$ and $xP_i''y$ and, for all $i \notin K$, $zP_i''x$ and $yP_i''x$. Because xD_ky it follows that $xP_0''y$, and because $zP_i''x$ for all $i \in N$ it follows from weak Pareto that $zP_0''x$. Because R'' satisfies requirements (a), (b), and (c) of step one, it follows that $R_0''\big|_{\{x,y,z\}} \in \mathcal{R}^*\big|_{\{x,y,z\}}$ and therefore $zP_0''y$. By the independence of irrelevant alternatives axiom, this implies that $z\bar{D}_Ky$. In other words:

$$xD_Ky \text{ implies that } z\bar{D}_Ky. \tag{2}$$

By interchanging y and z in statement (2) it follows that:

$$xD_Kz \text{ implies that } y\bar{D}_Kz. \tag{3}$$

By replacing x by y, y by z, and z by x in statement (1) it follows that:

$$yD_Kz \text{ implies that } y\bar{D}_Kx. \tag{4}$$

By combining statements (1), (3), and (4) it follows that:

$$xD_Ky \text{ implies that } y\bar{D}_Kx. \tag{5}$$

By interchanging x and y in statements (1), (2), and (5) it follows that

$$\begin{aligned} &yD_Kx \text{ implies that } y\bar{D}_Kz,\\ &yD_Kx \text{ implies that } z\bar{D}_Kx,\\ &yD_Kx \text{ implies that } x\bar{D}_Ky, \end{aligned}$$

and therefore by combining statement (5) it follows that:

$$xD_Ky \text{ implies that } y\bar{D}_Kz, z\bar{D}_Kx, \text{ and } x\bar{D}_Ky. \tag{6}$$

Therefore, we are led to the implication that:

$$\text{for every } \{x,y,z\} \subseteq X, \text{ if } xD_Ky \text{ then } a\bar{D}_Kb \text{ for every } a,b \in \{x,y,z\}. \tag{7}$$

Clearly, statement (7) applies if we replace z with w. By replacing z with w in statement (1) it follows that

$$xD_Ky \text{ implies that } x\bar{D}_Kw. \tag{8}$$

By replacing y with w in statement (2) it follows that

$$xD_Kw \text{ implies that } z\bar{D}_Kw. \tag{9}$$

By replacing x with z and y with w in statement (5) it follows that

$$zD_Kw \text{ implies that } w\bar{D}_Kz. \tag{10}$$

By combining statements (7), (8), (9), and (10), we are led to the result that, for every $\{x, y\}, \{w, z\} \subseteq X$, if xD_Ky then $w\bar{D}_Kz$. This concludes the proof. □

Proof of Theorem 1. We assume that the rationality measure satisfies properties 1, $2'$, and $3'$. This will be sufficient to prove the theorem.

Let f be a collective choice rule that satisfies the monotonicity, weak Pareto, independence of irrelevant alternatives, and non-dictatorship axioms. We will derive a contradiction.

Let $S \subseteq N$ be coalition of minimal size, so that $|T| < |S|$ implies that xD_Ty is false for all $x, y \in X$. By the weak Pareto axiom, such a coalition S exists. By the non-dictatorship axiom and Lemma 1, $|S| \geq 2$. Without loss of generality, let xD_Sy. Let $S_1 \subseteq S$ such that $|S_1| = 1$, let $S_2 \equiv S \setminus S_1$, and let $S_3 \equiv N \setminus S$.

Let $R \in \mathcal{R}^{*N}$ be a transitive profile such that (a) xP_iy and yP_iz for all $i \in S_1$, (b) zP_ix and xP_iy for all $i \in S_2$, and (c) yP_iz and zP_ix for all $i \in S_3$. Let $R_* \in \mathcal{R}$ such that xP_*y, yP_*z, and zP_*x, and let $R_+ \in \mathcal{R}$ such that xP_+z, zP_+y, and yP_+x. Let $R^A, R^B, R^C \in \mathcal{R}^N$ be profiles such that (a) $R_i^A = R_i^B = R_i^C = R_*$ for all $i \in S_1$, (b) $R_i^A = R_i^B = R_i^C = R_+$ for all $i \in S_2$, and (c) $R_i^A = R_*$, $R_i^B = R_+$, and $R_i^C = R_i$ for all $i \in S_3$.

Because of the independence of irrelevant alternatives axiom, we can assume, without loss of generality, that the elements $x, y, z \in X$ are top-ranked in profiles R, R^A, R^B, and R^C and that $R\big|_{X\setminus\{x,y,z\}} = R^A\big|_{X\setminus\{x,y,z\}} = R^B\big|_{X\setminus\{x,y,z\}} = R^C\big|_{X\setminus\{x,y,z\}}$. It follows that $R \succeq R^A$, $R \succeq R^B$, and $R \succeq R^C$. Therefore, by monotonicity, if one or more of R_0^A, R_0^B, and R_0^C is transitive, then R_0 must be transitive.

Suppose, contrariwise, that R_0 is not transitive. It follows that neither R_0^A, R_0^B, nor R_0^C may be transitive. Because S_2 is not a decisive coalition, it follows that xR_0^Ay, yR_0^Az, and zR_0^Ax. Because R_0^A is not transitive it follows that $S_1 \cup S_3$ must be decisive for at least one of the three pairs x over y, y over z, or z over x. By Lemma 1, it follows that $xD_{S_1\cup S_3}y$ for all $x, y \in X$.

Because S_1 is not a decisive coalition, it follows that xR_0^Bz, zR_0^By, and yR_0^Bx. Because R_0^B is not transitive it follows that $S_2 \cup S_3$ must be decisive for at least one of the three pairs x over z, z over y, or y over x. By Lemma 1, it follows that $xD_{S_2\cup S_3}y$ for all $x, y \in X$.

Because $xD_{S_1\cup S_3}y$ for all $x, y \in X$ it follows that yP_0^Cz, zP_0^Cx. Because $xD_{S_2\cup S_3}y$ for all $x, y \in X$ it follows that yP_0^Cx. Therefore it follows that R_0^C is transitive, which is a contradiction, proving that R_0 must be transitive.

By assumption, the coalition $S = S_1 \cup S_2$ is decisive for x over y. This implies that xP_0y. Because zP_iy only for $i \in S_2$ and S_2 is not decisive, it follows that yR_0z. Because R_0 is transitive, it follows that xP_0z. But this means that $xD_{S_1}z$, which implies, by Lemma 1, that S_1 is a dictator. This violates the non-dictatorship axiom, and concludes the impossibility proof.

Independence of the Axioms. We describe four collective choice rules. Each of the rules satisfies three of the axioms while violating the fourth. This is sufficient to prove the independence of the axioms.

Rule 1. For all $x, y \in X$, let xR_0y if and only if $|\{i \in N : xR_iy\}| \geq |\{i \in N : yR_ix\}|$. This rule clearly satisfies weak Pareto, independence of irrelevant alternatives, and non dictatorship, but violates monotonicity.

Rule 2. Let $d \in N$. For all $x, y \in X$, let xR_0y if and only if xR_dy. This rule clearly satisfies monotonicity, weak Pareto, and independence of irrelevant alternatives, but violates non-dictatorship.

Rule 3. Let $\mathcal{R}^T$ be the set of preference relations such that $R \in \mathcal{R}^T$ and $R' \succeq R$ implies that $R' \in \mathcal{R}^T$. If $R_1, R_2 \in \mathcal{R}^T$, let $f(R_1, ..., R_n) = R_1$, otherwise, let $f(R_1, ..., R_n) = R_2$. This rule satisfies monotonicity, weak Pareto, and non dictatorship, but violates independence of irrelevant alternatives.

Rule 4. For all $x, y \in X$, let xR_0y. This rule clearly satisfies monotonicity, independence of irrelevant alternatives, and nondictatorship, but violates weak Pareto. □

References

Arrow, K. J. (1963): *Social Choice and Individual Values*, Yale University Press, second ed.

Barthelemy, J.-P. (1982): "Arrow's theorem: Unusual domains and extended codomains," *Mathematical Social Sciences*, 79–89.

Batra, R. and K. Pattanaik (1972): "Transitive multi-stage majority decisions with quasi-transitive individual preferences," *Econometrica*, 40, 1121–1135.

Black, D. (1948a): "The Decisions of a Committee Using a Special Majority," *Econometrica*, 16, 245–261.

——— (1948b): "On the Rationale of Group Decision-Making," *Journal of Political Economy*, 56, 23–34.

Blair, D. H. and R. A. Pollak (1982): "Acyclic Collective Choice Rules," *Econometrica*, 50, 931–943.

Bossert, W. and K. Suzumura (2007): "Domain Closedness Conditions and Rational Choice," *Order*, 24, 75–88.

Cato, S. (2011): "Social choice without the Pareto principle: a comprehensive analysis," *Soc Choice Welf.*

Cherepanov, V., T. Feddersen, and A. Sandroni (2008): "Rationalization," Mimeo, Northwestern University.

Dasgupta, P. and E. Maskin (2008): "On the Robustness of Majority Rule," *Journal of the European Economic Association*, 6, 949–973.

Gibbard, A. (1969): "Social Choice and the Arrow Conditions," Unpublished manuscript.

Inada, K. (1969): "The simple majority decision rule," *Econometrica*, 37, 490–506.

Manzini, P. and M. Mariotti (2007): "Sequentially rationalizable choice," *American Economic Review*, 97, 1824–1839.

Mas-Colell, A. and H. Sonnenschein (1972): "General Possibility Theorems for Group Decisions," *Review of Economic Studies*, 39, 185–192.

Masatlioglu, Y., D. Nakajima, and E. Ozbay (forthcoming): "Revealed attention," *American Economic Review.*

Monjardet, B. (1978): "An Axiomatic Theory of Tournament Aggregation," *Mathematics of Operations Research*, 3, 334–351.

Rubinstein, A. and Y. Salant (forthcoming): "Eliciting welfare preferences from behavioural data sets," *Review of Economic Studies.*

Schick, F. (1969): "Arrow's Proof and the Logic of Preference," *Philosophy of Science*, 36, 127–144.

Sen, A. (1969): "Quasi-Transitivity, Rational Choice and Collective Decisions," *Review of Economic Studies*, 36, 381–393.

——— (1977): "Social Choice Theory: A Re-Examination," *Econometrica*, 45, 53–88.

Sen, A. K. (1964): "Preferences, Votes and the Transitivity of Majority Decisions," *Review of Economic Studies*, 31, 163–165.

——— (1966): "A possibility theorem on majority decisions," *Econometrica*, 34, 491–499.

——— (1970): *Collective Choice and Social Welfare*, Holden-Day, Inc.

Suzumura, K. (1983): *Rational choice, collective decisions, and social welfare*, Cambridge University Press.

Tversky, A. (1969): "Intransitivity of preferences," *Psychological Review*, 76, 31–48.