Inefficiency Measurement

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Abstract

We introduce an ordinal model of efficiency measurement. Our primitive is a notion of efficiency that is comparative, but not cardinal or absolute. In this framework, we postulate axioms that we believe an ordinal efficiency measure should satisfy. Primary among these are choice consistency and planning consistency, which guide the measurement of efficiency in a firm with access to multiple technologies. Other axioms include scale-invariance, which states that pounds and kilograms are treated the same, strong monotonicity, which states that efficiency should decrease if the inputs and outputs remain static when the technology becomes unambiguously more efficient, and a very mild continuity condition. These axioms characterize a family of path-based measures. By replacing the continuity condition with symmetry, which states that the names of commodities do not matter, we obtain the coefficient of resource utilization.

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1 Introduction

Since the beginning of economics as a science, economists have tried to address the fundamental question of how to measure the efficiency of economic systems. A classical answer to this problem was provided by Debreu (1951), who introduced a simple method to measure the underutilization of resources called the *coefficient of resource utilization*. Debreu's coefficient has enjoyed a very rich history in applied economics, primarily as a result of its operationalization for applied economists by Farrell (1957). See, for example, Nishimizu and Page (1982), Blomström (1986), Färe, Grosskopf, and Lovell (1985), Färe, Grosskopf, and Lovell (1994), or Färe, Grosskopf, Norris, and Zhang (1994).

Our contribution here is twofold. First, we suggest an explicitly ordinal framework for the study of efficiency measurement. This framework allows us to study efficiency measurement using an axiomatic approach without resorting to an ad-hoc cardinal benchmark. Secondly, using two properties of efficiency measures, we offer an entirely new ordinal characterization of a family of efficiency measures, which includes the coefficient as a special case. This family need not treat all commodities symmetrically, unlike the coefficient. This is the family of *path-based rules*, which we describe below. Finally, in an ordinal exercise similar to that of Christensen, Hougaard, and Keiding (1999), we also show that the coefficient emerges as the unique symmetric path-based rule. This exercise applies very broadly, but for reasons of concreteness we focus on the case of production.

We understand efficiency measurement as the problem of comparing efficiency across different production possibility sets. Economists are routinely faced with the problem of judging how efficiently one firm performs compared to another firm. Or, a firm may be concerned with how efficiently one plant performs compared to another plant. In our conception, efficiency involves two factors: the firm's *technology*—the production possibilities available to that firm—and the choices of inputs and outputs made by that firm. Thus we study the efficiency of the chosen input/output combination *relative* to the given technology.

For a given technology, there is a set of resource bundles which could be utilized without hurting production, say P. We term P the *input set*. The resource bundle that is actually used, say x, may or may not be efficient for this input set. An *efficiency measure* is a ranking of these pairs of objects, enabling comparisons of the form: resource bundle x is more efficient for input set P than is resource bundle y for input set Q.[1] This primitive can also be found in Hougaard and Keiding (1998) and Christensen, Hougaard, and Keiding (1999) in a cardinal form, so that a ranking is not posited, but rather is a functional representation.

The ultimate purpose of these measures is to determine which economic unit (out of a feasible set) performs the most efficiently, and is not to say *how* efficient a given unit is. For this reason, efficiency measures in our model are ordinal (or comparative) and not cardinal (or absolute). A cardinal measure of efficiency can be constructed easily by applying the ordinal measure to some specific benchmark.

When we speak of an axiom being "ordinal" as opposed to "cardinal," we mean simply that the axiom references only the comparative order structure and not the details of the functional form. Any ordinal axiom can be phrased in

[1] Our framework therefore discusses what is typically called *input efficiency*. This modeling choice postulates an implicit independence axiom: the fact that the efficiency measure depends on the input set and not on the technology as a whole bears some relation to independence axioms found in social choice, most notably the independence of irrelevant alternatives axioms of Arrow (1963) and Nash (1950).

cardinal terms, but the converse is only sometimes true. While we are the first to explicitly model the problem as an ordinal one and stress the importance of the ordinal approach in efficiency measurement, previous works by Hougaard and Keiding (1998) and Christensen, Hougaard, and Keiding (1999) involve axioms which can, for the most part, be phrased in ordinal terms. (The primary exception is the existence of a functional representation.) In this sense, these works are an important predecessor to ours and are probably the first to use an implicit ordinal approach.

The ordinal approach suggests several natural axioms on efficiency measures which we believe have not been described before in an ordinal setting. These axioms, which we refer to as planning and choice consistency, guide the measurement of efficiency in a firm with access to multiple technologies. Planning consistency describes a natural way to measure efficiency in the case of a firm which must commit to producing without knowing which relevant technology will be feasible tomorrow. Choice consistency describes a natural way to measure efficiency in the case of a firm which must choose one, and only one, technology from which to produce. Versions of these axioms are found in Hougaard and Keiding (1998) and Christensen, Hougaard, and Keiding (1999) in cardinal form.

From these two axioms, we derive several results. The first is a characterization of a class of rules which can be viewed as "generalized" numeraire rules. These rules, which we call *path-based measures*, work as follows. Each such measure is associated with a fixed and monotonic continuous path emanating from the origin and ending at some fixed point. For any pair of inputs and outputs, we scale the path so that the end of the path coincides with the vector of inputs, and then find the point of intersection of the path with the input set associated with that level of output. The further along this path, the more efficient the bundle of inputs. The characterization of path-based measures relies on three additional axioms: scale invariance, strong monotonicity, and *monotone continuity*, a basic continuity condition related to axioms found in the decision theory literature. (See Arrow (1971).)

The *scale invariance* axiom has a natural interpretation: no matter which unit of measurement we choose, the measure will result in the same outcome. To illustrate the *strong monotonicity* axiom, suppose that we have two technologies, P and P'. We can say that it is unambiguously more efficient to produce an output under P' than under P if for any input x which can produce the output under P, it is possible to produce the same output under P' using strictly fewer of all resources contained in x. If technology becomes unambiguously more efficient yet we retain the previous level of inputs and outputs, then the axiom requires that there should be a strict decrease in the measured efficiency of the firm.

With cardinal versions of our axioms, Christensen, Hougaard, and Keiding (1999) characterize the coefficient of resource utilization along with a symmetry axiom. To this end, we describe such a symmetry axiom in our environment and establish an ordinal variant of their result. Our result differs from Christensen, Hougaard, and Keiding (1999) in a few technical respects. First, our characterization removes two of their axioms (continuity on rays and dominance). Second, we do not assume existence of a functional representation. Third, our theorem also applies on the domain of convex problems, while the Christensen, Hougaard, and Keiding (1999) result requires the existence of non-convex technologies. Lastly, their finite union property and conditional multiplicity axioms are somewhat weaker than choice consistency and planning consistency in that the former only apply in the case of indifference.

1.1 Related Literature

1.1.1 Previous axiomatic work on efficiency measurement

We study the framework of technical efficiency measures, as introduced by Farrell (1957). Debreu (1951) envisioned his coefficient as being applied to the efficiency of an economy as a whole, involving both production and consumption. Obviously, all of our results apply to this environment with a suitable reinterpretation of our axioms.[2] In Debreu's setting, efficiency is understood in the Pareto sense. It is, of course, also possible to think of the case in which efficiency is determined according to a social welfare function. Along these lines, Graaff (1977) calculates Debreu's coefficient (a) with respect to the Scitovsky set, and (b) with respect to the curve from Samuelsonian aggregation that leads us to the specified welfare level, and then takes the ratio of the latter to the former.[3] (See de Scitovszky (1942) and Samuelson (1956).)

Previous axiomatic work on efficiency measurement generally takes a given technology as primitive, notable exceptions being Hougaard and Keiding (1998) and Christensen, Hougaard, and Keiding (1999). An efficiency measure operates with respect to that prespecified technology. (See, for example, Färe and Lovell (1978) and Russell (1985).) In contrast, our approach specifies an efficiency measure which can work *across* technologies. The setup of our framework postulates an implicit independence axiom (only input sets matter). This amounts to an assumption that our measure is really a measure of *input* efficiency. A dual approach might study measures of output efficiency. To some degree, we require such a framework as our interpretation of technology

[2]To extend the exercise to Debreu's framework, it would be necessary to add an independence assumption requiring that two economies with the same Scitovsky set be treated identically. Such an axiom first appears in Chambers and Hayashi (2012).

[3]For a discussion of Graaff's index, see Fleurbaey (2009).

may be different from preceding works. Our definition conceives of technology of the specific resources available to a given firm at a given point in time; it is the classical notion of a production possibility set. Other such definitions seek to understand whether society is operating at an efficient level, given the current state of the art (in a general equilibrium context, this would be the Minkowski sum of all individual production sets, as in Debreu (1959)).

Several existing axiomatic characterizations of efficiency measures presuppose the existence of a numeraire by which one can measure efficiency. (See, for example, Färe and Lovell (1978); Russell and Schworm (2009, 2011).) Following Luenberger (1992, 1996), Chambers, Chung, and Färe (1996) introduce a class of line-based measures. These lines also pass through the current inputs, but differ from the coefficient of resource utilization in that their gradient is determined by a numeraire which may reflect prices or intrinsic value. The family of path-based measures may be seen as a generalization of this class.

Probably the closest work to ours is the axiomatic contribution of Christensen, Hougaard, and Keiding (1999). These authors introduce a cardinal framework which otherwise is very similar to ours. Our purpose has been to understand the "correct" departures from the coefficient of resource utilization in asymmetric environments, so a natural building block is their work which characterizes the coefficient. We offer a counterpart of their theorem in our ordinal framework (our Theorem 2) in order to highlight the connection. We also show how such a characterization can be established on the domain of convex sets.

1.1.2 Path-based measures

Aside from the aforementioned contributions of Debreu (1951) and Farrell (1957), the idea of using a path to compare alternatives relative to some set is not new, and seems to date back at least as far as Dupuit (1844). The classical reason for studying these objects was in order to cardinally measure changes in welfare. When comparing two consumption bundles, one can find the indifference set on which the second bundle lies, and then take a path-based measure based on the original consumption. The welfare change in such a measure is determined by the distance one would need to travel on this path. The paths considered in this literature were typically straight lines following an axis—effectively measuring utility using a numeraire.

Wold (1943a,b, 1944) illustrates a classical construction of utility functions (taught in most current economics textbooks) based on following a path from the origin and finding the point in which this path intersects a specific indifference curve. Allais (1952, 1981) suggests path-based rules as a method of defining welfare change (analogous to compensating or equivalent variation). Luenberger (1992, 1996) also discusses generalized path-based rules as welfare measures.

Social choice and Nash bargaining theory are rife with path-based style rules. In particular, Kalai (1977), Thomson and Myerson (1980), Bossert, Nosal, and Sadanand (1996), and Kıbrıs and Tapkı (2010) axiomatize bargaining rules based on monotone paths. (See also Moulin (1988, page 81).)

Nevertheless, as far as we can tell, our characterization of path-based measures is new.

1.1.3 Mathematics and lattice homomorphisms

Formally, our two axioms, planning and choice consistency, imply that rules (for a fixed vector of inputs) are *lattice homomorphisms*, from a certain lattice of subsets (ordered by set inclusion) to the lattice of real numbers (with the typical ordering). Kreps (1979) seems to be the first to state an axiom analogous to choice consistency, albeit in an entirely different framework. He observed already that this axiom was necessary and sufficient (in a finite world) for a binary relation over sets to be generated by maximization of another binary relation over points. An analogue of this result plays an implicit role in the proof of our own result, and is the driving force behind the characterization of Hougaard and Keiding (1998), who derive necessary and sufficient conditions for an efficiency measure to be characterized by the minimization of a function (normalized by inputs) on the input set.

Miller (2008), Chambers and Miller (2011), Leclerc and Monjardet (2011), Leclerc (2011) and Dimitrov, Marchant, and Mishra (2012) study variations of the planning and choice consistency axioms in other economic environments.

2 The model and results

A set $X \in \mathbb{R}^\ell$ is *comprehensive*[4] if, for all $x, y \in \mathbb{R}^\ell$, $x \in X$ and $y \geq x$ implies that $y \in X$.[5] Let Σ denote the set of comprehensive and closed sets $P \subseteq \mathbb{R}^\ell_+$. A set $P \in \Sigma$ is referred to an *input set*. Let $\mathcal{P} \subseteq \Sigma \times \mathbb{R}^\ell_{++}$ be such that $(P, x) \in \mathcal{P}$ only if $x \in P$. An ordered pair $(P, x) \in \mathcal{P}$ is referred to as an

[4]Comprehensivity refers to free disposability. Free disposability may be unduly strong in the case of possible congestion effects. A generalization of this concept is that of "ray monotonicity," see Färe, Grosskopf, and Lovell (1987). Our measures continue to satisfy all of the postulated axioms when input sets are only required to be ray monotonic.

[5]Vector inequalities: $x \geq y$ if $x_i \geq y_i$ for all i, $x > y$ if $x \geq y$ and $x \neq y$, and $x \gg y$ if $x_i > y_i$ for all i.

efficiency measurement problem. In particular, we are interested in studying two domains of efficiency measurement problems. First, let $\mathcal{P}' \subseteq \Sigma \times \mathbb{R}^{\ell}_{++}$ denote the *general domain*, for which $(P, x) \in \mathcal{P}'$ if and only if $x \in P$. Second, let $\mathcal{P}'' \subseteq \mathcal{P}'$ denote the *domain of convex problems*, for which $(P, x) \in \mathcal{P}''$ if and only if $(P, x) \in \mathcal{P}'$ and P is convex.

The two domains we consider are each historically of interest to economists. For example, convex technologies are particularly of interest in a general equilibrium setting. The proof structure of our characterization results depends on the domain of interest.

An *ordinal efficiency measure* is a binary relation $\succeq$ on $\mathcal{P}$. We discuss several properties of ordinal efficiency measures.

The first axiom is standard: it merely states that the ranking should be complete and transitive. This axiom rules out the Malmquist Index and other non-transitive measures. (See Färe, Grosskopf, and Roos (1997).)

Weak order: The binary relation $\succeq$ is complete and transitive.

The second axiom was described in the introduction. It relates to a firm that must commit to producing without knowing which relevant technology will be feasible tomorrow.

Planning consistency: For all P, Q and all $x \in P \cap Q$, if $(P, x) \succeq (Q, x)$, then $(P \cap Q, x) \sim (P, x)$.

The third axiom, also described in the introduction, relates to a firm that has the option of choosing one, and only one, technology from which to produce. Note that this axiom is stated in a nonbinary fashion (that is, it refers to arbitrary finite collections P_i). This is so because on the domain of convex problems, it is not necessarily the case that $\bigcup_i P_i$ is a feasible input set.

Choice consistency: For all x and all finite collections P_i for which $x \in P_i$, if $(\bigcup_i P_i, x) \in \mathcal{P}$ and $(P_i, x) \succeq (P_j, x)$ for all i, then $(\bigcup_i P_i, x) \sim (P_j, x)$.

Choice consistency is equivalent to the following on the general domain of problems. The proof is a simple induction argument.

Weak choice consistency: For all P, Q and all $x \in P \cap Q$, if $(P, x) \succeq (Q, x)$, then $(P \cup Q, x) \sim (Q, x)$.

The next property, strong monotonicity, states that as technology becomes unambiguously better, then remaining at current production levels must be considered worse. Note that a weak version of monotonicity is already implied by either planning or choice consistency.

Strong monotonicity: If $P \subseteq \text{int}Q$, then $(P, x) \succ (Q, x)$.[6]

For the next axiom, we need some basic definitions. For every $\lambda \in \mathbb{R}^\ell_{++}$ and $x \in \mathbb{R}^\ell$, define $\lambda * x \equiv (\lambda_1 x_1, ..., \lambda_\ell x_\ell)$. Similarly, $\lambda * P = \{\lambda * x : x \in P\}$. Scale invariance states that pounds and kilograms are treated the same by the measure.

Scale invariance: For all $(P, x) \in \mathcal{P}$ and all $\lambda \in \mathbb{R}^\ell_{++}$, $(P, x) \sim (\lambda * P, \lambda * x)$.

Lastly, we introduce a basic continuity axiom.

Monotone Continuity: Let $\{F_i\}_{i \in \mathbb{N}} \subseteq \Sigma$ be a decreasing sequence of sets for which $\bigcap_{i \in \mathbb{N}} F_i \in \Sigma$. Let $E \in \Sigma$ and $x \in \bigcap_{i \in \mathbb{N}} F_i \cap E$. If $(E, x) \succeq (F_i, x)$ for all $i \in \mathbb{N}$, then $(E, x) \succeq (\bigcap_{i \in \mathbb{N}} F_i, x)$.

[6] When we refer to interior, we mean the relative interior with respect to $\mathbb{R}^\ell_+$.

An *increasing path* is defined as a continuous mapping $g : [0, 1] \to \mathbb{R}^\ell$ for which $g(0) = \mathbf{0}$, $g(1) = \mathbf{1}$ and for which $x > y$ implies $g(x) > g(y)$. A scale-invariant path-based measure is one for which there exists an increasing path g such that, for all $(P, x) \in \mathcal{P}$, $f(P, x) = \inf\{\beta : g(\beta) * x \in P\}$. An ordinal efficiency measure is *path-based* if there exists a scale-invariant path-based measure f such that $(P, x) \succeq (Q, y)$ if and only if $f(P, x) \geq f(Q, y)$.

Theorem 1. *On either the general domain or the domain of convex problems, a ordinal efficiency measure satisfies the weak order, planning consistency, choice consistency, monotone continuity, strong monotonicity and scale invariance axioms if and only if it is path-based. Furthermore, the six axioms are independent.*

We illustrate the set of path-based measures by means an example which we believe to be new to this literature: the *lexicographic commodity ranking.* Suppose that the commodities are prioritized in terms of "importance," so that commodity 1 is more important than commodity 2, and so forth. Consider the path $g : [0, 1] \to \mathbb{R}^\ell$ given by $g_i(x) = 0$ if $x \leq \frac{i-1}{\ell}$, $g_i(x) = \ell \cdot (x - \frac{i-1}{\ell})$ if $\frac{i-1}{\ell} < x \leq \frac{i}{\ell}$, and $g_i(x) = 1$ if $\frac{i}{\ell} < x$.

The ordinal efficiency measure associated with this path compares two problems by the proportion of commodity ℓ that could be reduced without hurting production. If, for both problems, commodity ℓ could be eliminated without hurting production, the measure proceeds by comparing the proportion of commodity $\ell - 1$ that could be reduced without hurting production, and so forth.

Each ordering of the commodities implies a different rule. If we take the expectation with respect to all such paths (according to a uniform measure), the resulting path corresponds to the coefficient of resource utilization. By changing the weighting of the lexicographic orderings, we can generate a rule which incorporates different tradeoffs in the prioritization of different commodities.

2.1 The Coefficient of Resource Utilization

The coefficient of resource utilization (Debreu, 1951) is a path-based measure associated with the straight line from the current inputs to the origin. To characterize this axiom we introduce a strong axiom, symmetry, which states that all commodities should be treated equally according to the measure. It forbids us from giving precedence to one commodity over another in terms of efficiency measurement.

For every permutation $\sigma : \{1, ..., \ell\} \to \{1, ..., \ell\}$, define $\sigma \circ x \equiv (x_{\sigma(1)}, ..., x_{\sigma(\ell)})$. Similarly, $\sigma \circ P = \{\sigma \circ x : x \in P\}$.

Symmetry: For all $(P, x) \in \mathcal{P}$ and all permutations σ, $(P, x) \sim (\sigma \circ P, \sigma \circ x)$.

The *coefficient of resource utilization* of Debreu (1951) is the function $f_c : \mathcal{P} \to [0, 1]$ given by $f_c(P, x) \equiv \inf\{\alpha : \alpha x \in P\}$.

Theorem 2. *There is a unique ordinal efficiency measure satisfying the axioms weak order, planning consistency, choice consistency, strong monotonicity, scale invariance, and symmetry on either the general domain or the domain of convex problems. It is represented by the coefficient of resource utilization; that is*

$$(P, x) \succeq (Q, y) \text{ if and only if } f_c(P, x) \geq f_c(Q, y). \tag{1}$$

Furthermore, the six axioms are independent.

Monotone continuity is not required for this result. Consequently, any ordinal efficiency measure which satisfies either of the monotone continuity or symmetry axioms in addition to the other five must be a path-based measure.

3 Other domains

Certain environments may allow the negative production of certain commodities. For example, firms may be able to borrow against future production. In such an environment, it is reasonable to suppose that the set of possible input sets $\hat{\Sigma}$ consist of all comprehensive, nonempty, and closed sets $P \subseteq \mathbb{R}^\ell$ which are bounded below in the sense that there is a point $x \in \mathbb{R}^\ell$ for which $x \leq y$ for all $y \in P$.[7] In this case, we can define our domain $\hat{\mathcal{P}}$ to include all problems (P, x) for which $P \in \hat{\Sigma}$ and $x \in P$.

Most of the axioms we previously described can be immediately described in this environment. We need to modify strong monotoncity, so that the interior operator is interpreted as the usual interior (not the relative interior). Scale invariance needs to be removed altogether. Instead, we suggest replacing scale invariance with the following axiom, which we label *translation invariance*. For $y \in \mathbb{R}^\ell$, we define $P + y \equiv \{x \in \mathbb{R}^\ell : x - y \in P\}$.

Translation invariance: For all $P, Q \in \hat{\Sigma}$, all $x \in P \cap Q$, and all $y \in \mathbb{R}^\ell$,

$(P, x) \succeq (Q, x)$ if and only if $(P + y, x + y) \succeq (Q + y, x + y)$.

Because the zero production of all commodities is no longer a lower bound in this setting, translation invariance requires that the location of the origin not matter when determining the efficiency of an economic system.

It is now easy to describe a counterpart of the family described in Theorem 1. We define an *unbounded increasing path* to be a continuous function $g : (-\infty, 0] \to \mathbb{R}^\ell$ which is strictly increasing, and satisfying $g(0) = 0$, and finally, for all $\alpha \in \mathbb{R}_+$, there exists $x \in (-\infty, 0]$ for which $\|g(x)\| = \alpha$. The last requirement simply states that the path is "unbounded below."

[7] One may think of this boundedness property as a borrowing constraint in negative productions.

We can now define an analogue of the path-based measures. A translation-invariant path-based measure $f : \hat{\mathcal{P}} \to (-\infty, 0]$ is one for which there exists an unbounded increasing path g such that $f(P, x) = \inf\{\beta \leq 0 : g(\beta) + x \in P\}$. For example, the measures described by Chambers, Chung, and Färe (1996) are translation-invariant path-based measures whereby $g(\beta) = b\beta$, for some fixed $b \in \mathbb{R}^{\ell}_{+}$.

An ordinal efficiency measure on $\hat{\mathcal{P}}$ is *translation invariant path-based* if it there exists a translation-invariant path-based measure f such that $(P, x) \succeq (Q, y)$ if and only if $f(P, x) \geq f(Q, y)$.[8]

The following is the analogue of Theorem 1 where scale-invariance is replaced by translation-invariance. We state it without proof, as the details are almost identical to the proof of Theorem 1.

Theorem 3. *On $\hat{\mathcal{P}}$, an ordinal efficiency measure satisfies the weak order, planning consistency, choice consistency, monotone continuity, strong monotonicity and translation invariance axioms if and only if it is translation invariant path-based. Furthermore, the six axioms are independent.*

4 Conclusion

This paper has introduced the notion of an ordinal efficiency measure. We have suggested that the ultimate interest of efficiency measurement is to compare alternative production plans. The comparative structure of such a problem suggests an ordinal approach, rather than a cardinal one. By so doing, we have been able to generate a large class of rules for measuring efficiency which depart from classical rules. The utility of these rules lies in their freedom to

[8] An ordinal efficiency measure on $\hat{\mathcal{P}}$ is simply an ordinal ranking on $\hat{\mathcal{P}}$, just as it was for the domain $\mathcal{P}$.

adjust efficiency measurement in order to accommodate tradeoffs between a lexicographic approach, and an approach where all commodities are treated equally (the coefficient of resource utilization).

Appendix

The following axiom and lemma will be used in several of the proofs.

Monotonicity: If $P \subseteq Q$, then $(P, x) \succeq (Q, x)$.

Monotonicity is implied by the planning consistency axiom. The following lemma is straightforward and will be stated without proof.

Lemma 1. *If $\succeq$ satisfies planning consistency then $\succeq$ satisfies monotonicity.*

The next lemma is used in the proofs of both theorems.

Lemma 2. *If $\succeq$ satisfies scale-invariance and weak order, and f is a scale-invariant path-based measure, then the following two statements are equivalent:*

1. $(P, x) \succeq (Q, y)$ if and only if $f(P, x) \geq f(Q, y)$ for all $(P, x), (Q, y) \in \mathcal{P}$.

2. $(P, \mathbf{1}) \succeq (Q, \mathbf{1})$ if and only if $f(P, \mathbf{1}) \geq f(Q, \mathbf{1})$ for all $(P, \mathbf{1}), (Q, \mathbf{1}) \in \mathcal{P}$.

Proof. For $x \in \mathbb{R}^{\ell}_{++}$, let x^{-1} be the inverse, so that $x^{-1} * x \equiv \mathbf{1}$. By scale invariance, $(P, x) \sim (x^{-1} * P, \mathbf{1})$. Let $P^x \equiv x^{-1} * P$ and let $Q^y \equiv y^{-1} * Q$. By transitivity, $(P, x) \succeq (Q, y)$ if and only if $(P^x, \mathbf{1}) \succeq (Q^y, \mathbf{1})$. For any scale-invariant path-based measure there exists an increasing path g such that $f(P, x) = \inf\{\beta : g(\beta) * x \in P\}$. Clearly $g(\beta) * x \in P$ if and only if $g(\beta) \in P^x$. Therefore, $f(P, x) \geq f(Q, y)$ if and only if $f(P^x, \mathbf{1}) \geq f(Q^y, \mathbf{1})$. □

To prove Theorem 1 we make use of the following proposition.

Proposition 1. *Suppose that $\succeq$ satisfies planning consistency and monotone continuity. Then for every $(P,x) \in \mathcal{P}$, there exists some $Q \subseteq \mathbb{R}^\ell_+$ such that $(P,x) \succeq (P',x)$ if and only if $Q \subseteq P'$.*

Proof. Let $\succeq$ satisfy the axioms and let $(P,x) \in \mathcal{P}$. Define $\{Q_\lambda\}_{\lambda\in\Lambda} \equiv \{R : (P,x) \succeq (R,x)\}$ and define $Q \equiv \bigcap_{\lambda\in\Lambda} Q_\lambda$. Clearly, if $(P,x) \succeq (P',x)$ then $Q \subseteq P'$. Let $(P',x) \in \mathcal{P}$ for which $Q \subseteq P'$. We claim there exists a sequence $\{Q_i\}_{i\in\mathbb{N}} \subseteq \{Q_\lambda\}_{\lambda\in\Lambda}$ satisfying $\bigcap_{i\in\mathbb{N}} Q_i = Q$. To see this, suppose this claim is false, let D be a countable dense subset of $\mathbb{R}^n$, and let $D^* \equiv D \setminus Q$. Write $D^* = \{z_n : n \in \mathbb{N}\}$, and for each n choose $\lambda(n) \in \Lambda$ such that $z_n \notin Q_{\lambda(n)}$. Let $Q' \equiv \bigcap_{n\in\mathbb{N}} Q_{\lambda(n)}$. By construction, $Q = \bigcap_{\lambda\in\Lambda} Q_\lambda \subseteq Q'$. Let $z \in Q' \setminus Q$. Then there is an open set G containing z such that $G \cap Q = \varnothing$, and G must contain elements of D^*, a contradiction which proves the claim.[9] By planning consistency, we may without loss of generality choose this sequence to be decreasing with respect to set inclusion. By monotonicity, $(P,x) \succeq (Q_i,x) \succeq (Q_i \cup P', x)$ for all $i \in \mathbb{N}$. By monotone continuity, $(P,x) \succeq (\bigcap_{i=1}^\infty (Q_i \cup P'), x)$. Because $\bigcap_{i=1}^\infty (Q_i \cup P') = Q \cup P' = P'$ it follows that $(P,x) \succeq (P',x)$. □

Proof of Theorem 1

It is straightforward to verify that path-based measures satisfy the six axioms. Here we prove the converse. Let $\succeq$ satisfy the six axioms. We show that there exists a scale-invariant path-based measure f such that $(P,x) \succeq (Q,y)$ if and only if $f(P,x) \geq f(Q,y)$.

By Lemma 2, it is sufficient to show that $(P,\mathbf{1}) \succeq (Q,\mathbf{1})$ if and only if $f(P,\mathbf{1}) \geq f(Q,\mathbf{1})$. To simplify the proof we will write (P) in place of $(P,\mathbf{1})$.

Let $(P) \in \mathcal{P}$. By Proposition 1 there exists $Q \subseteq \mathbb{R}^\ell_+$ such that $(P) \succeq (P')$

[9] We thank an anonymous referee for suggesting this simple proof.

if and only if $Q \subseteq P'$. By monotonicity, $(Q) \sim (P)$. Therefore, for any (P'), $(Q) \succeq (P')$ if and only if $Q \subseteq P'$. For $S \subseteq \mathbb{R}^\ell$, define $\mathcal{C}(S) \equiv \bigcup_{x \in S}\{y \in \mathbb{R}^\ell : x \leq y\}$.

We claim that $x^* \equiv \bigwedge\{x : x \in Q\} \in Q$, and consequently that $Q = \mathcal{C}(\{x^*\})$. Assume, contrariwise, that this is false. Because Q is closed and $x^* \notin Q$ there exists $x' \gg x^*$ such that $x' \notin Q$. Because $x' \gg x^*$, there exists a set of ℓ points $x^1, ..., x^\ell \in Q$ such that, for all $i \leq \ell$, $x'_i \gg x^i_i \gg x^*_i$. Define $H_i = \{x \in \mathbb{R}^\ell_+ : x_i \geq x'_i\}$. Let $i \leq \ell$. Because (a) $\mathbf{1} \in Q \cap H_i$, (b) $Q \cap H_i \in \Sigma$, and (c) $Q \cap H_i$ is convex whenever Q is convex, it follows that $(Q \cap H_i) \in \mathcal{P}$.[10] Next, because $x^i \notin H_i$, it follows that $Q \not\subseteq Q \cap H_i$, and therefore that $(Q) \not\succeq (Q \cap H_i)$. From weak order it follows that $(Q \cap H_i) \succ (Q)$. From choice consistency it follows that $(\bigcup_{i=1}^\ell Q \cap H_i) \succ (Q)$. It is easily verified that $Q \subseteq \bigcup_{i=1}^\ell H_i$, consequently, $(Q) \succ (Q)$, a contradiction.

Thus for all $(P) \in \mathcal{P}$ there exists $L(P) \in \mathbb{R}^\ell_+$, $L(P) \leq \mathbf{1}$, such that $(P) \succeq (Q)$ for all $(Q) \in \mathcal{P}$ such that $L(P) \in Q$. Define $G \equiv \bigcup_{(P) \in \mathcal{P}} L(P)$. Clearly, $G \subseteq [0,1]^\ell$. For all $x, y \in G$, either $x \geq y$ or $y \geq x$. Otherwise, $(\mathcal{C}(\{x\})) \in \mathcal{P}$ and $(\mathcal{C}(\{y\})) \in \mathcal{P}$ would be unordered, violating weak order.

Next, we show that there is a function g that satisfies the conditions of the theorem. For $\beta \in [0,1]$, let $H(\beta) \equiv \{x \in \mathbb{R}^\ell_+ : \sum_{i=1}^\ell x_i = \ell\beta\}$. Let $g(\beta) \equiv G \cap H(\beta)$. Because $G \subseteq [0,1]^\ell \subseteq \bigcup_\beta H(\beta)$ it follows that $g([0,1]) = G$. The monotonicity of g follows from the fact that G is totally ordered. For fixed β, the points in $H(\beta)$ are unordered with respect to $\leq$; thus $g(\beta)$ is at most single-valued. To show that it is single-valued, let $\beta \in [0,1)$ such that $g(\beta) = \varnothing$ and we will derive a contradiction. Let $K(\beta) \equiv \mathcal{C}(H(\beta))$, and note that $\alpha > \beta$ implies that $K(\alpha) \subseteq \text{int}K(\beta)$. Because $g(\beta) = \varnothing$, it follows that

[10]Condition (c) guarantees that the proof applies to the domain of convex problems in addition to the general domain.

$L(K(\beta)) \in \text{int}K(\beta)$. Let β' be such that $g(\beta') = L(K(\beta))$. Then $K(\beta') \in \text{int}K(\beta)$. By strong monotonicity, this implies that $(K(\beta')) \succ (K(\beta))$. But $L(K(\beta)) \in K(\beta')$ implies, by the definition of L, that $(K(\beta)) \succeq (K(\beta'))$, a contradiction. Clearly, $L(\mathcal{C}(\{\mathbf{1}\})) = \mathbf{1}$ thus $g(1) = \mathbf{1}$. By strong monotonicity, $Q \succ \mathcal{C}(\{\mathbf{0}\})$ for all $Q \neq \mathcal{C}(\{\mathbf{0}\})$ thus $g(0) = \mathbf{0}$. It is easy to see that g is continuous.[11]

Lastly, we show that $(P) \succeq (Q)$ if and only if $\inf\{\beta : g(\beta) \in P\} \geq \inf\{\beta : g(\beta) \in Q\}$. First, assume that $(P) \succeq (Q)$. Let $\beta^* = \inf\{\beta : g(\beta) \in P\}$. Because g is continuous, $g(\beta^*) \in P$. We claim that $L(P) = g(\beta^*)$. Suppose, contrariwise, that $L(P) \neq g(\beta^*)$. By definition, $g(\beta^*) = L(R)$ for some $R \in \Sigma$. Because $L(R) \in P$, $(R) \succeq (P)$. Because $L(P) \in \mathcal{C}(\{L(P)\})$, $(P) \succeq (\mathcal{C}(\{L(P)\}))$ and therefore $(R) \succeq (\mathcal{C}(\{L(P)\}))$. Thus $L(R) \in \mathcal{C}(\{L(P)\})$, a contradiction. Thus $L(P) = g(\beta^*)$. Because $L(P) \in Q$ it follows by comprehensiveness that $\{\beta : g(\beta) \in P\} \subseteq \{\beta : g(\beta) \in Q\}$. To prove the other direction, let $\inf\{\beta : g(\beta) \in P\} \geq \inf\{\beta : g(\beta) \in Q\}$. By the definition, $L(P) \in \{\beta : g(\beta) \in P\}$ and by comprehensiveness $\{\beta : g(\beta) \in P\} \subseteq \{\beta : g(\beta) \in Q\}$. Thus $L(P) \in \{\beta : g(\beta) \in Q\}$ which implies that $(P) \succeq (Q)$.

That the axioms are independent is proven below.

Proof of Theorem 2

It is clear that the measure represented by f_c satisfies these properties. Let $\succeq$ satisfy the six axioms, and let $(P, x), (Q, y) \in \mathcal{P}$ be efficiency measurement problems. We show that statement (1) must hold. This proves uniqueness.

[11] To see that the path is continuous, let $\beta_k \to \beta^*$ be a sequence. We want to show that $g(\beta_k) \to g(\beta^*)$. Let U be a neighborhood of $g(\beta^*)$. Because U is a neighborhood of $g(\beta^*)$, there exists a sufficiently small $\varepsilon > 0$ such that if $|x_i - g_i(\beta^*)| < \varepsilon$ for all i, then $x \in U$. Now consider the interval $(\beta^* - \varepsilon, \beta^* + \varepsilon)$. By the monotonicity of g, for any $\beta \in (\beta^* - \varepsilon, \beta^* + \varepsilon)$, we know that $|g_i(\beta) - g_i(\beta^*)| < \varepsilon$.

By Lemma 2, because f_c is a scale-invariant path-based measure, it is sufficient to show that $(P, \mathbf{1}) \succeq (Q, \mathbf{1})$ if and only if $f_c(P, \mathbf{1}) \geq f_c(Q, \mathbf{1})$. To simplify the proof we will write (P) in place of $(P, \mathbf{1})$.

For $\beta \in [0, 1]$, define $\mathcal{K}(\beta) \equiv \{x \in \mathbb{R}^\ell_+ : x_i \geq \beta \text{ for all } i\}$. For $P \in \Sigma$, let $\beta(P) \equiv \inf\{\beta : \mathcal{K}(\beta) \subseteq P\}$. Clearly $f_c(P) = \beta(P)$. By strong monotonicity $\beta(P) \geq \beta(Q)$ if and only if $(\mathcal{K}(\beta(P))) \succeq (\mathcal{K}(\beta(Q)))$. We show that for any problem (P), $(P) \sim (\mathcal{K}(\beta(P)))$. This implies (by transitivity) that $(P) \succeq (Q)$ if and only if $(\mathcal{K}(\beta(P))) \succeq (\mathcal{K}(\beta(Q)))$, proving statement (1). There are two cases.

The General Domain: For $\beta \in [0, 1]$, let $\mathcal{K}_i(\beta) \equiv \{x \in \mathbb{R}^\ell_+ : x_i \geq \beta\}$. Then $\bigcap_{i=1}^{\ell} \mathcal{K}_i(\beta) = \mathcal{K}(\beta)$. For $j > 1$, let σ_{1j} be the permutation such that $\sigma_{1j}(1) = j$, $\sigma_{1j}(j) = 1$, and $\sigma_{1j}(k) = k$ for $k \neq 1, j$. Note that $\sigma_{1j} \circ \mathcal{K}_1(\beta) = \mathcal{K}_j(\beta)$. By symmetry, $(\mathcal{K}_1(\beta)) \sim (\sigma_{1j} \circ \mathcal{K}_1(\beta))$ and therefore $(\mathcal{K}_1(\beta)) \sim (\mathcal{K}_j(\beta))$ for all $j > 1$. By planning consistency and an induction argument, $(\mathcal{K}_1(\beta)) \sim (\bigcap_{i=1}^{k} \mathcal{K}_i(\beta))$ for every $k \leq \ell$ and thus $(\mathcal{K}_1(\beta)) \sim (\mathcal{K}(\beta))$. By choice consistency, $(\bigcup_{i=1}^{\ell} \mathcal{K}_i(\beta)) \sim (\mathcal{K}_1(\beta))$. Therefore $(\bigcup_{i=1}^{\ell} \mathcal{K}_i(\beta)) \sim (\mathcal{K}(\beta))$. Let $(P) \in \mathcal{P}'$. For $P \in \Sigma$, P is closed and comprehensive, and thus $\mathcal{K}(\beta(P)) \subseteq P$. By construction of the sets $\mathcal{K}_i$ it follows that $P \subseteq \bigcup_{i=1}^{\ell} \mathcal{K}_i(\beta(P))$. By monotonicity, $(P) \sim (\mathcal{K}(\beta(P)))$.

The Domain of Convex Problems: For $\beta \in [0, 1]$, let $R(\beta) \equiv \{x : \sum_{i=1}^{\ell} x_i \geq \ell\beta\}$, and let $R_i(\beta) \equiv R(\beta) \cap \mathcal{K}_i(\beta)$. Note that (a) $\sigma_{1j} \circ R_1(\beta) = R_j(\beta)$ for all $j > 1$, (b) $\bigcap_{i=1}^{\ell} R_i(\beta) = \mathcal{K}(\beta)$, and (c) $\bigcup_{i=1}^{\ell} R_i(\beta) = R(\beta)$. By symmetry, $(R_1(\beta)) \sim (\sigma_{1j} \circ R_1(\beta))$ and thus $(R_1(\beta)) \sim (R_j(\beta))$ for all $j > 1$. By planning consistency and an induction argument, $(R_1(\beta)) \sim (\bigcap_{i=1}^{k} R_i(\beta))$ for every $k \leq \ell$ and therefore $(R_1(\beta)) \sim (\mathcal{K}(\beta))$. By choice consistency, $(\bigcup_{i=1}^{\ell} R_i(\beta)) \sim (R_1(\beta))$, so that $(R(\beta)) \sim (\mathcal{K}(\beta))$. Let $(P) \in \mathcal{P}''$. By symmetry, $(P) \sim (\bigcap_{\sigma}(\sigma \circ P))$. For $P \in \Sigma$, $\mathcal{K}(\beta(P)) \subseteq \bigcap_{\sigma}(\sigma \circ P) \subseteq R(\beta(P))$. By monotonicity,

$(P) \sim (\bigcap_\sigma(\sigma \circ P)) \sim (\mathcal{K}(\beta(P)))$.

That the axioms are independent is proven below.

Independence of the Axioms

We present six measures. Five violate the named axiom while satisfying the other six. The sixth violates symmetry and monotone continuity while satisfying the other five. This proves that both sets of axioms are independent. For all but the first measure, let $(P, x) \succeq (Q, y)$ if and only if $f_k(P, x) \geq f_k(Q, y)$.

Weak order: For $x = y$, let $(P, x) \succeq (Q, y)$ if and only if $f_c(P, x) \geq f_c(Q, y)$. For $x \neq y$, let $(P, x) \sim (Q, y)$ if and only if (i) $(P, x) = (\lambda * Q, \lambda * y)$ for some $\lambda \in \mathbb{R}^\ell_{++}$ or (ii) $(P, x) = (\sigma \circ Q, \sigma \circ y)$ for some permutation σ.

Planning consistency: Let $f_2(P, x) \equiv 1 - \max\left\{\beta \leq 1 : \{y \in \mathbb{R}^\ell_+ : \frac{\|y-x\|}{\|x\|} \leq \beta\} \subseteq P\right\}$.

Choice consistency: Let $f_3(P, x) \equiv \max\left\{\beta \leq 1 : \{y \in \mathbb{R}^\ell_+ : \frac{\|y\|}{\|x\|} \leq \beta\} \not\subseteq P\right\}$.

Strong monotonicity: Let $f_4(P, x) \equiv 1$.

Scale invariance: Let $f_5(P, x) \equiv \inf\{\alpha : \alpha\mathbf{1} \in P\}$.

Monotone continuity and Symmetry: Let $g : [0, 1] \to \mathbb{R}^\ell$ be defined by $g_i(x) = 0$ if $x \leq \frac{i-1}{\ell}$, $g_i(x) = \ell \cdot (x - \frac{i-1}{\ell})$ if $\frac{i-1}{\ell} < x \leq \frac{i}{\ell}$, and $g_i(x) = 1$ if $\frac{i}{\ell} < x$. Let $f_6(P, x) = \sup\{\alpha : x * g(\alpha) \notin \text{int}P\}$.

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