Group Identification

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Comments welcome.

Abstract

I study a model of group identification in which individuals' opinions as to the membership of a group are aggregated to form a list of group members. Potential aggregation rules are studied through the axiomatic approach. I introduce two axioms, *meet separability* and *join separability*, each of which requires the list of members generated by the aggregation rule to be independent of whether the question of membership in a group is separated into questions of membership in two other groups. I use these axioms to characterize a class of *one-vote* rules, in which one opinion determines whether an individual is considered to be a member of a group. I then show that the only anonymous one-vote rule is *self-identification*, in which each individual determines for himself whether he is a member of the group.

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1 Introduction

In 1997, the White House decided that, for purposes of Federal data collection, you are African-American if you claim you are.[1] In addition, the White House decided that you can be both African-American and White. These policy changes were first implemented in the 2000 decennial census.

The policymakers charged with revising the policy were guided by several principles. Two of these principles stand out. First, they rejected the view that race can be objectively defined. Second, they desired that results be comparable across Federal agencies with different needs. They chose a standard set of racial categories and allowed agencies to use additional categories "provided they can be aggregated to" the standard ones. Other principles included minimizing cost, respecting individual dignity, and the understanding that "the standards are not intended to be used to establish eligibility for participation in any federal program."

Increases in the "number of persons born who are of mixed race or ethnicity" led to the relatively uncontroversial decision to allow individuals to be counted as members of multiple racial groups. The decision to use self-identification was more controversial. Some Federal agencies were concerned that changes in the method of data collection could make it difficult to study historical trends. However, other methods were also known to have problems in this regard and could lead to an individual being identified differently among data sets.

I argue that the decision to switch to self-identification was appropriate in light of the policymakers' concerns. To study this question I set forth a model of group identification in which individuals are classified into groups on the basis of opinions. I introduce several properties which reflect the policymakers' concerns and show that self-identification is the only data collection method that satisfies these properties.

I follow a model of group identification first introduced by Kasher and Rubinstein (1997). Individuals have opinions as to which members of society are members of a particular group. A *social rule* is a systematic method for aggregating opinions of agents. Opinions and outcomes are binary. Each individual is either believed (or determined) to be a member of the group, or not. Social rules are studied through an axiomatic approach: various properties are proposed and rules satisfying these properties are characterized.

I extend the Kasher-Rubinstein model by introducing multiple groups with predefined relationships. Federal statistical policy envisions many racial groups, including Asians, Whites, people who are members of either group ("Asian or White"), and people who are members of both ("Asian and White"). The set of groups also includes the non-members of each group (e.g. "Non-Whites" and "Non-Asians").

Opinions must preserve these relationships. A person who believes that his neighbor is "Asian and White" must also believe that his neighbor is Asian. A social rule takes the group label as an argument, so that it is allowed to work differently for different groups. It need not aggregate opinions about Asians in the same way that it aggregates opinions about Whites. However, a consequence of the axioms described below is that a rule must be independent of the group under consideration.

[1] Statistical Policy Directive No. 15, Race and Ethnic Standards for Federal Statistics and Administrative Reporting, 62 FR 58782, October 30, 1997.

The focus of this paper is to understand the implications of a property called *separability*, which requires that social rules preserve certain relationships between groups. I introduce two axioms which implement different aspects of this concept.

Meet separability requires social rules to yield the same list of "Asian and White" people regardless of which of two possible approaches is used. The first approach is to aggregate opinions about "Asian and White" people. The second approach is to generate two lists by aggregating opinions about Asians and about Whites separately, and then to take the intersection of the two lists.

Join separability is a similar, except that it is defined with respect to disjunction instead of conjunction. It requires social rules to yield the same list of "Asian or White" people regardless of which of two possible approaches is used. The first approach is to aggregate opinions about "Asian or White" people. The second approach is to generate two lists by aggregating opinions about Asians and about Whites separately, and then to take the union of the two lists.

In addition to the separability axioms, I require *non-degeneracy.* This axiom requires that the opinions be relevant in determining whether each person is a member of a group. There are cases when this axiom is not appropriate. To find the group of people whose height exceeds six feet, we do not need opinions. Two yardsticks are sufficient. In the absence of objective standards, opinions are needed to classify people into groups.

Using these three axioms I characterize a family of rules called *one-vote* rules. These rules associate with each person a single opinion that determines whether that person is *qualified* (determined to be a member of the group).

An *anonymity* axiom requires that the qualification of individuals does not depend on their names. Anonymity is appealing because government policymakers cannot arbitrarily favor some people above others. Non-governmental researchers gathering data may not be bound by this normative requirement, but may lack an external scientific basis to draw distinctions between individuals. I show that self-identification is the only anonymous one-vote rule.

The problem of group identification extends beyond the sphere of Federal agencies trying to classify individuals for statistical purposes. Legal systems allocate rights and restrictions to groups, whether these be particular minority groups or the group of licensed drivers. Philosophers try to understand whether groups exist. Sociologists analyze social groups, anthropologists examine cultures, political scientists study collective action,[2] and historians use groups to explain past human behavior. In economics there has been a significant increase in the use of group membership as a variable in formal modeling following the recent work of Akerlof and Kranton (2000).

1.1 Related Literature

The model of group identification was first introduced by Kasher and Rubinstein (1997). Their paper included a characterization of self-identification as the only rule satisfying symmetry, a weak independence condition, and the "liberal principle," which required that individuals can force certain outcomes.[3] Much of the notation in this paper was introduced

[2] See Olson (1971).

[3] This follows a refinement of the Kasher-Rubinstein characterization by Sung and Dimitrov (2005).

by Samet and Schmeidler (2003), who studied a family of consent rules characterized by anonymity, monotonicity, and a strong independence condition. They also characterize self-identification as the only rule satisfying monotonicity, non-degeneracy, the stronger independence condition, and another property labeled self-determination.

Nearly every paper which considers the Kasher-Rubinstein model studies social rules which satisfy an *independence* axiom.[4] The stronger version of this axiom, found in Kasher and Rubinstein (1997), Samet and Schmeidler (2003), Ju (2005), and Çengelci and Sanver (2006), requires that whether a particular individual is determined to be a member of a group is independent of the opinions regarding all of the other individuals. A weaker version of this axiom, found in Kasher and Rubinstein (1997), Sung and Dimitrov (2005), and Dimitrov et al. (2007),[5] allows an individual's status to be affected by opinions about other individuals if some other individual's status is also affected. This paper departs completely from the requirement of independence.

The Kasher-Rubinstein framework is applicable in determining which individuals meet a particular standard, such as the set of students eligible for merit scholarships. A related but conceptually distinct problem involves ranking individuals according to a standard. For example, a school might wish to create a ranking of students. The latter problem has been studied axiomatically by Palacios-Huerta and Volij (2004) in the context of developing a cardinal ranking of scientific publications.

[4]An exception is Houy (2006) which classifies individuals as members of a group if they are "indirectly designated by all the individuals in the society."

[5]Using the weaker independence condition, Dimitrov et al. (2007) characterize a recursive procedure for determining group membership.

2 The Model

2.1 The model and the notation

I extend the model introduced by Kasher and Rubinstein (1997) and use the notation introduced by Samet and Schmeidler (2003). There is a set $N \equiv \{1, ..., n\}$ of individuals, $n \geq 3$.[6] There is a set of groups, $\mathscr{G}$, that forms a Boolean algebra under conjunction, disjunction, and negation. Each element $a \in \mathscr{G}$ describes membership in a group. For example, if $a, w \in \mathscr{G}$ are the groups "Asian" and "White," respectively, then $\mathscr{G}$ also contains the groups $a \wedge w$ ("Asian and White"), $a \vee w$ ("Asian or White") and $\bar{a}$ ("Non-Asian"). The collection $\mathscr{G}$ also contains a minimal group $\mathtt{0}$ ("no one") and a maximal group $\mathtt{1}$ ("everyone"). I denote by $\mathcal{G} \equiv \mathscr{G} \setminus \{\mathtt{0}, \mathtt{1}\}$ this set minus these minimal and maximal groups.

A categorical **view** about a group is an N-vector, the j-th component of which is 1 if individual j is viewed as a member, and 0 otherwise. The set of views is denoted by $\mathcal{V} \equiv \{0, 1\}^N$. A **profile** is a vector of views $P = (P_1, ..., P_n) \in \mathcal{V}^N$ where P_i represents individual i's view. I write P_{ij} to denote individual i's opinion about individual j. A **qualification problem** is a pair $(P, a) \in \mathcal{V}^N \times \mathcal{G}$. A **social rule** is a function $f : \mathcal{V}^N \times \mathcal{G} \to \mathcal{V}$ which associates each qualification problem with a **social opinion** $f(P, a) \equiv (f_1(P, a), ..., f_n(P, a))$.

As usual, for a set of the form $\{0, 1\}^I$, I write $x \geq y$ if $x_i \geq y_i$ for all $i \in I$, $x > y$ if $x \geq y$ and $x \neq y$, and $(\bar{x}_i) = (1 - x_i)$. I define the **meet** ($\wedge$) as the coordinatewise minimum, so that $(x \wedge y)_i = \min\{x_i, y_i\}$, and the **join** ($\vee$) as the coordinatewise maximum, so that $(x \vee y)_i = \max\{x_i, y_i\}$. Lastly, I denote $\mathbf{0}$ and $\mathbf{1}$ as the elements of the set composed entirely of zeros and ones, respectively.

For any two qualification problems (P, a) and (Q, b), I define $(P, a) \wedge (Q, b) \equiv (P \wedge Q, a \wedge b)$, $(P, a) \vee (Q, b) \equiv (P \vee Q, a \vee b)$, and $\overline{(P, a)} \equiv (\bar{P}, \bar{a})$. A direct implication of this definition is that the opinions with respect to groups $a \wedge \bar{a} = \mathtt{0}$ and $a \vee \bar{a} = \mathtt{1}$ are represented by $(P, a) \wedge \overline{(P, a)} = (\mathbf{0}, \mathtt{0})$ and $(P, a) \vee \overline{(P, a)} = (\mathbf{1}, \mathtt{1})$. Everyone believes that no one is a member of the group "no one," and that everyone is a member of the group "everyone." I define $f(\mathbf{0}, \mathtt{0}) \equiv \mathbf{0}$ and $f(\mathbf{1}, \mathtt{1}) \equiv \mathbf{1}$.

A set of qualification problems is **consistent** if, for any two qualification problems (P, a) and (Q, b) in the set, the following three properties hold.

1. When a and b encompass the entire set (e.g., "Asian or White" and "Non-Asian"), then everyone should believe that everyone is a member of a or b.
   - If $a \vee b = \mathtt{1}$, then $(P, a) \vee (Q, b) = (\mathbf{1}, \mathtt{1})$.
2. When a and b are mutually exclusive (e.g., "Asian and White" and "Non-White"), then everyone should believe that no one is a member of both.
   - If $a \wedge b = \mathtt{0}$, then $(P, a) \wedge (Q, b) = (\mathbf{0}, \mathtt{0})$.
3. When a includes b (e.g., "White" includes "Asian and White"), then everyone should believe that members of b are also members of a.
   - If $a \vee b = a$, then $(P, a) \vee (Q, b) = (P, a)$.

I denote by $\mathscr{C}$ the set of consistent two-element sets of qualification problems.

[6] With the exception of Theorem 2.6, all results would hold if I allowed the case where n=2.

2.2 The axioms

Let a and w be the groups of Asians and Whites, respectively, and let (A, a) and (W, w) describe the opinions about these groups. Then $a \wedge w$ is the group of "Asian and White" people and $(A, a) \wedge (W, w) \equiv (A \wedge W, a \wedge w)$ describes the opinions about that group.

There are two ways to generate a list of "Asian and White" people. The *single ballot approach* will directly generate a list of "Asian and White" people: $f((A, a) \wedge (W, w))$. The *two ballot approach* will generate two lists; one of Asians, $f(A, a)$, and one of Whites, $f(W, w)$. One can then generate a list of "Asian and White" people by taking the names common to both lists. This is the meet of the two lists: $f(A, a) \wedge f(W, w)$. The first axiom, *meet separability*, requires that these lists be the same.

Meet separability: For every consistent set of qualification problems $\{(P, a), (Q, b)\} \in \mathscr{C}$, $f((P, a) \wedge (Q, b)) = f(P, a) \wedge f(Q, b)$.

Similarly, $a \vee w$ is the group of "Asian or White" people and $(A, a) \vee (W, w) \equiv (A \vee W, a \vee w)$ describes the opinions about that group.

There are two ways to generate a list of "Asian or White" people. The *single ballot approach* will directly generate a list of "Asian or White" people: $f((A, a) \vee (W, w))$. The *two ballot approach* will generate two lists; one of Asians, $f(A, a)$, and one of Whites, $f(W, w)$. One can then generate a list of "Asian or White" people by taking the names which appear on either list. This is the join of the two lists: $f(A, a) \vee f(W, w)$. The second axiom, *join separability*, requires that these lists be the same.

Join separability: For every consistent set of qualification problems $\{(P, a), (Q, b)\} \in \mathscr{C}$, $f((P, a) \vee (Q, b)) = f(P, a) \vee f(Q, b)$.

The third axiom is adapted from Samet and Schmeidler (2003). This axiom excludes constant rules — rules for which there exists an individual who is, or is not, a member of the group regardless of which names are on the ballots.

Non-degeneracy: For every individual j and every group $a \in \mathcal{G}$ there exist profiles P, P' such that $f_j(P, a) = 1$ and $f_j(P', a) = 0$.

Each of the separability axioms implies *monotonicity*, which requires that no names be removed from the list of qualified persons as additional names are added to the ballots.

Monotonicity: For every group $a \in \mathcal{G}$, $P \geq P'$ implies that $f(P, a) \geq f(P', a)$.

The proof of the following lemma is straightforward and is left for readers.

Lemma 2.1. *If a social rule f satisfies either of the meet separability or join separability axioms then it satisfies monotonicity.*

Social rules satisfy *group independence* if they use the same method to aggregate opinions about every group.

Group independence: For all groups $a, b \in \mathcal{G}$ and every profile P, $f(P, a) = f(P, b)$.

None of these axioms directly requires group independence. One might use one method to aggregate opinions about Asians, a different method to aggregate opinions about Whites, and a third method to aggregate opinions about "Asians and Whites". However, rules which satisfy non-degeneracy and either separability axiom necessarily satisfy group independence, as I show in the following proposition.[7] Every rule discussed in the paper satisfies this property. To simplify the notation I will sometimes drop the group label and write "$f(P)$" in place of "$f(P, a)$ for every $a \in \mathcal{G}$".

Proposition 2.2. *If a social rule f satisfies the non-degeneracy axiom and either of the meet separability or join separability axioms then it satisfies group independence.*

Proof. Non-degeneracy and monotonicity directly imply that $f(\mathbf{1}, a) = \mathbf{1}$ for all $a \in \mathcal{G}$.

Let the pair $\{(P, a), (\mathbf{1}, b)\} \in \mathscr{C}$ such that $(*)$ $a \neq a \wedge b \neq b$. By meet separability, $f(P, a) \wedge f(\mathbf{1}, b) = f(P \wedge \mathbf{1}, a \wedge b)$ and thus $f(P, a) = f(P, a \wedge b)$. From $(*)$ it follows that the pair $((\mathbf{1}, a), (P, b))$ is consistent, and thus $f(P, b) = f(P, a \wedge b)$. Therefore $f(P, a) = f(P, b)$. For every $a, b \in \mathcal{G}$ such that $(*)$ does not hold there exists an element $c \in \mathcal{G}$ such that $a \neq a \wedge c \neq c \neq c \wedge b \neq b$ and thus $f(P, a) = f(P, b)$ for all $a, b \in \mathcal{G}$.

The second half of the proof is the dual of the first and can be proved by replacing "meet" ($\wedge$) with "join" ($\vee$) and 1 with 0. $\square$

2.3 The main characterizations

I now define three families of rules which are characterized by combinations of axioms from the preceding subsection. In each family rules associate with each individual a non-empty set of relevant opinions which uniquely determine whether the individual is qualified. The families differ by the degree of cohesiveness that the relevant opinions must demonstrate for the individual to be qualified, as well as by the size of the relevant set. The relevant opinions are neither required to be about nor otherwise related to the individual with whom the set is associated.

2.3.1 Agreement Rules

The first such family of rules are **agreement** rules. An individual is qualified as a member of a group if every opinion in the relevant set is in favor of qualification. These rules can equivalently be defined by associating with each individual a minimal profile P in which $P_{ij} = 1$ if and only if i's opinion about j is relevant.[8] This family of rules is characterized by meet separability and non-degeneracy.

Agreement rules: For every individual j there exists a profile $P^{j-} > \mathbf{0}$ such that, for all groups $a \in \mathcal{G}$, $f_j(P, a) = 1$ if and only if $P \geq P^{j-}$.

These rules are characterized in the following theorem:

[7] To understand why non-degeneracy is necessary, consider any rule where $f(P, a) = f(P, a \wedge b)$ and $f(P, b) = \mathbf{1}$. This rule satisfies meet separability but neither non-degeneracy nor group independence.

[8] P is the minimal profile which leads to qualification. Note that the minimal profile cannot be $\mathbf{0}$; otherwise the set of relevant opinions would be empty.

Theorem 2.3. *A social rule f satisfies the meet separability and non-degeneracy axioms if and only if it is an agreement rule. Moreover, both axioms are independent.*

Proof. By Proposition 2.2 rules satisfying meet separability and non-degeneracy are group independent. Let $P, Q \in \mathcal{V}^N$ and $j \in N$. Define $\mathcal{P}_j \equiv \{P \in \mathcal{V}^N : f_j(P) = 1\}$. We know that $\mathcal{P}_j \neq \varnothing$ by the non-degeneracy axiom.

Define $P^{j-} \equiv \wedge_{P \in \mathcal{P}_j} P$. For all profiles $P', P'' \in \mathcal{P}_j$, $f_j(P') = f_j(P'') = 1$. By the meet separability axiom, $f_j(P' \wedge P'') = 1$. It follows by an induction argument that $f_j(\wedge_{P \in \mathcal{P}_j} P) = f_j(P^{j-}) = 1$. Therefore, $P^{j-} \in \mathcal{P}_j$.

Clearly, for all profiles $P \in \mathcal{P}_j$, $P \geq \wedge_{P \in \mathcal{P}_j} P = P^{j-}$. Furthermore, $P^{j-} \neq \mathbf{0}$, otherwise $f_j(P) = 1$ for all profiles P, which would violate the non-degeneracy axiom.

Lastly, I show that for all profiles P such that $P \geq P^{j-}$, $P \in \mathcal{P}_j$. Let $P \geq P^{j-}$. By monotonicity, $f_j(P) \geq f_j(P^{j-})$. Because $f_j(P^{j-}) = 1$ it follows that $P \in \mathcal{P}_j$. Hence $P \in \mathcal{P}_j$ if and only if $P \geq P^{j-}$. It follows that, for all $a \in \mathcal{G}$, $f_j(P, a) = 1$ if and only if $P \geq P^{j-}$.

The independence of the axioms is proved in the appendix. □

2.3.2 Nomination Rules

The second family of rules are **nomination** rules. An individual is qualified as a member of a group if any opinion in the relevant set is in favor of qualification. These rules can equivalently be defined by associating with each individual a maximal profile P in which $P_{ij} = 0$ if and only if i's opinion about j is relevant.[9] This family of rules is characterized by join separability and non-degeneracy.

Nomination rules: For every individual j there exists a profile $P^{j+} < \mathbf{1}$ such that, for all groups $a \in \mathcal{G}$, $f_j(P, a) = 0$ if and only if $P \leq P^{j+}$.

These rules are characterized in the following theorem:

Theorem 2.4. *A social rule f satisfies the join separability and non-degeneracy axioms if and only if it is a nomination rule. Moreover, both axioms are independent.*

Proof. This is the dual of Theorem 2.3 and can be proved by exchanging 0 and 1 and replacing "meet" ($\wedge$) with "join" ($\vee$), $\geq$ with $\leq$, and P^{j-} with P^{j+}.

The independence of the axioms is proved in the appendix. □

2.3.3 One-Vote Rules

The third family of rules are **one-vote** rules, for which the relevant set associated with each individual consists of a single opinion. The individual is qualified if that opinion is in favor of qualification. One-vote rules are characterized by meet separability, join separability, and non-degeneracy.

One-vote rules: For every individual j there exists (i, k) in $N \times N$ such that, for all groups $a \in \mathcal{G}$, $f_j(P, a) = P_{ik}$.

[9] P is the maximal profile leading to disqualification. Note that the maximal profile cannot be $\mathbf{1}$; otherwise the set of relevant opinions would be empty.

From this follows the main result:

Theorem 2.5. *A social rule f satisfies the meet separability, join separability, and non-degeneracy axioms if and only if it is a one-vote rule. Moreover, all three axioms are independent.*

Proof. That the one-vote rules satisfy the three axioms is trivial. I show that any social rule that satisfies the three axioms is necessarily a one-vote rule. Let $j \in N$. Because f satisfies meet separability and non-degeneracy it must be an agreement rule (by Theorem 2.3). Therefore, there must exist a profile $P^{j-} > \mathbf{0}$ such that $f_j(P) = 1$ if and only if $P \geq P^{j-}$. This implies that there exists (i, k) in $N \times N$ such that $P^{j-}_{ik} = 1$. It follows that if $P_{ik} = 0$ then $f_j(P) = 0$ and therefore $P_{ik} \geq f_j(P)$.

Because f satisfies join separability and non-degeneracy it must be an agreement rule (by Theorem 2.4). Therefore, there must exist a profile $P^{j+} < \mathbf{1}$ such that $f_j(P) = 0$ if and only if $P \leq P^{j+}$. Let $P^* \in \mathcal{V}^N$ such that all elements are 1 except that $P^*_{ik} = 0$. We know that if $P^*_{ik} = 0$ then $f_j(P^*) = 0$ and therefore $P^* \leq P^{j+}$. Because $P^{j+} < \mathbf{1}$ it follows that $P^* = P^{j+}$ and thus $P^{j+}_{ik} = 0$. This implies that if $P_{ik} = 1$ then $f_j(P) = 1$ and therefore $f_j(P) \geq P_{ik}$. It follows that $f_j(P, a) = P_{ik}$ for every group $a \in \mathcal{G}$.

The independence of the axioms is proved in the appendix. □

2.4 Other Results

2.4.1 Self-identification

Kasher and Rubinstein (1997) introduced **self-identification**, in which each person decides whether to qualify herself.[10]

Self-identification: For every $j \in N$ and for every $a \in \mathcal{G}$, $f_j(P, a) = P_{jj}$.

The principle of equality of persons restricts an aggregation rule from making arbitrary distinctions among members of the society.[11] Samet and Schmeidler (2003) applied this principle through an **anonymity** condition which requires that the list of the qualified individuals does not depend on their names.[12] Names are switched through a permutation π of N. Thus, for a given permutation π, i is the new name of the individual formerly known as $\pi(i)$. For a given profile $P \in \mathcal{V}^N$, I let πP be the profile in which the names are switched. Then $(\pi P)_{ij} = P_{\pi(i)\pi(j)}$. I denote $\pi f(P, a) \equiv (f_{\pi(1)}(P, a), f_{\pi(2)}(P, a), ..., f_{\pi(n)}(P, a))$. The fourth axiom, anonymity, requires that if individual i is qualified in profile πP, then individual $\pi(i)$ is qualified in profile P.

[10] Self-identification is referred to as "the liberal rule" by Samet and Schmeidler and as the "strong liberal collective identity function" by Kasher and Rubinstein.

[11] In axiomatic economic theory this principle dates back at least as far as May (1951). This principle can be motivated either from a normative belief in equality such as that found in the writings of Locke and Jefferson, or from a positive concern that a researcher lacks a scientific basis upon which to draw such distinctions. Both motivations are relevant to government bureaucrats forbidden from arbitrarily preferring some people over others.

[12] Samet and Schmeidler call this condition symmetry. I have changed the name to minimize confusion with a different axiom of the same name introduced by Kasher and Rubinstein.

Anonymity: For every permutation π of N and every group $a \in \mathcal{G}$, $f(\pi P, a) = \pi f(P, a)$.

Self-identification is the only one-vote rule which satisfies the anonymity axiom.[13]

Theorem 2.6. *Self-identification is the only rule that satisfies the meet separability, join separability, non-degeneracy, and anonymity axioms. Moreover, all four axioms are independent.*

Proof. That self-identification satisfies the four axioms is trivial. I show that any rule that satisfies the four axioms must necessarily be self-identification. Let $j \in N$. Let f satisfy the meet separability, join separability, non-degeneracy, and anonymity axioms. By Theorem 2.5 f must be a one-vote rule, and therefore there must be a pair of individuals i and k such that $f_j(P, b) = P_{ik}$. Because the pair of individuals may differ for every individual j, I denote these individuals $i(j)$ and $k(j)$. Therefore, $f_j(P) = P_{i(j)k(j)}$. Let π be a permutation of N. Then, $f_j(\pi P) = (\pi P)_{i(j)k(j)} = P_{\pi(i(j))\pi(k(j))}$, and $f_{\pi(j)}(P) = P_{i(\pi(j))k(\pi(j))}$. By the anonymity axiom, it follows that $P_{\pi(i(j))\pi(k(j))} = P_{i(\pi(j))k(\pi(j))}$, which implies that $\pi(i(j)) = i(\pi(j))$ and $\pi(k(j)) = k(\pi(j))$, which hold if and only if $i(j) = k(j) = j$. Thus, for every individual $j \in N$ and every issue $a \in \mathcal{G}$, $f_j(P, a) = P_{jj}$.

The independence of the axioms is proved in the appendix. $\square$

2.4.2 Negation

Individuals in the model vote consistently: if (P, a) describes the opinions about group a, then $(\overline{P, a})$ describes the opinions about the group $\bar{a}$. Then $f(\overline{P, a})$ is the list of members of group $\bar{a}$ and the list of *non-members* of group a is given by $\overline{f(P, a)}$.

The fifth axiom, *negation*, requires that these two lists be the same.[14]

Negation: For all profiles P and all groups $a \in \mathcal{G}$, $\overline{f(P, a)} = f(\overline{P, a})$.

The negation axiom has another interpretation: it requires that social rules classify each person in the society as a member of a group or its complement but not both. The following proposition is necessary and sufficient to show that Theorems 2.5 and 2.6 remain true if one of the separability axioms is replaced by negation.

Proposition 2.7. *If a social rule f satisfies two of the meet separability, join separability, and negation axioms it satisfies the third.*

The proof is straightforward and is left to readers.

[13] A different approach to the principle of equality can be found in the *symmetry* condition of Kasher and Rubinstein (1997), which requires that if any two individuals are symmetric with respect to their views about others and others' views toward them, then either both or neither are qualified. Self-identification is not the only one-vote rule which satisfies the symmetry axiom; however, it is the only one-vote rule which satisfies symmetry and a stronger form of non-degeneracy. For more explanation and a proof see Miller (2006).

[14] Negation is related to the *self-duality* axiom introduced by Samet and Schmeidler. Both axioms require the social rule to preserve complementation and both are motivated by the view that one should be able to learn who is a member of a group by asking about the non-members of the group. Because of changes in the model, however, the axioms have rather different implications. Unlike self-duality, negation does not require a social rule to treat membership and non-membership in the same manner, nor does it require non-degeneracy.

3 Conclusion

I have extended the Kasher-Rubinstein model of group identification to allow social rules to aggregate opinions about different groups in different manners. I have introduced a concept of group separability and have shown that any non-degenerate rule satisfying both of the separability axioms is necessarily a one-vote rule, in which for each individual there is exactly one opinion which determines whether that person is qualified. The only anonymous one-vote rule is self-identification, under which each person determines for herself whether she is qualified. How we interpret these results depends on our understanding of the primitive.

The primary motivation set forth is to understand rules used to generate group data for research. This describes both Federal policy regarding data collection and the creation of data sets by social scientists. Here researchers decide that group identification should be a function of beliefs, either because the group cannot be objectively defined, or because the researchers lack other means to determine who is a member of a group. In this case it is important that seemingly trivial decisions made (whether to use one survey or two to generate the relevant data) should not have an unknown effect on the results of the research. Consequently, the results of this paper recommend the use of one-vote rules, especially when it is not clear how those later researchers will use the data. In cases where the researchers lack a basis for preferring some persons over others, self-identification should be used.

One-vote rules are also particularly nice because they require fewer opinions and therefore may be cheaper to generate. If the data set contains information on a small subset of S individuals out of a much larger society of N people, the one-vote rule requires the person creating the data set to seek out S opinions out of a total of N^2. Self-identification has other desirable properties. It is "liberal" in the sense that each person chooses whether she is a member of a group. Federal policymakers argued that it respects human dignity because it does "not tell an individual who he or she is, or specify how an individual should classify himself or herself."

Another possible understanding of the social rules studied in this paper is as a voting mechanism used to determine the composition of groups endowed with certain legal rights or obligations. The anonymity axiom seems desirable when allocating rights in a democratic society; consequently the separability axioms suggest the use of self-identification. However, it seems rather clear that individuals may have an incentive to distort their beliefs when group membership leads to a direct and tangible benefit or cost. Self-identification would not make much sense in this case. This tells us that sensible aggregation rules used to allocate legal rights will violate the separability axioms. An agenda setter may be able to influence the allocation of rights by dividing one question into two.

A third understanding is more philosophical. A popular view holds that a group is a social construct and only exists as a function of the beliefs about its composition. In this sense a social rule is part of the definition of a group. The separability axioms are very natural in this context because a given set of beliefs will always lead to a unique list of group members. The results of the paper suggest that there are limits on the method through which the beliefs can be aggregated. There cannot be groups defined by majority opinions, while there can be groups defined by self-inclusion. An alternate view is that a group is a social construct but that it exists as a function of beliefs other than the binary views considered in the model. It is impossible to evaluate this claim without adding more structure to the model. The case where opinions take the form of a totally ordered set is discussed next.

3.1 Generalizations of the Characterization Theorems

A possible extension to the group identification model would be to weaken the assumption that opinions about qualification are binary by replacing the domain of possible opinions from $\{0, 1\}$ to an arbitrary totally ordered set $\mathcal{D}$ containing minimal and maximal elements 0 and 1. Examples previously examined in the literature include the "trichotomous domain" studied by Ju (2005) in which $\mathcal{D} = \{0, \frac{1}{2}, 1\}$ (here $\frac{1}{2}$ has the meaning "no opinion"), and the unit interval $[0, 1]$ studied by Ballester and Garcia-Lapresta (2005). In this case profiles are elements of $\mathcal{D}^{N \times N}$. Social rules are mappings $f : \mathcal{D}^{N \times N} \times \mathcal{G} \to \mathcal{V}$. All of the axioms have natural analogues for this more general case.

If $\mathcal{D}$ is finite, then Theorems 2.3 and 2.4 remain unchanged, as the only assumption about $\mathcal{D}$ used in the proof is that $\mathcal{D}$ is finite. If $\mathcal{D}$ is not finite then neither of these theorems hold, as proofs rely on a finite induction argument. Proposition 2.2, however, is still applicable. In neither case does Theorem 2.5 hold; however, the rules characterized by meet separability, join separability, and non-degeneracy are very similar. As with one-vote rules, each individual is associated with a single relevant opinion. The individual is qualified if that opinion exceeds a cutoff point and is not qualified if the opinion is below that cutoff point. The rule additionally specifies whether the individual is qualified if that opinion is exactly at the cutoff point. For a definition of these rules and a proof see appendix B.

3.2 Weakening the axioms

A potential criticism of the result stems from the formulation of the axioms. One might object in that they govern relationships between irrelevant groups. For example, one might want a rule to aggregate opinions about British, Americans, and British-Americans consistently but not care about how the rule aggregates opinions about people who are either British or American. A weaker form of these axioms would apply only to pairs of groups in a subset of $\mathcal{G}$, where the subset is carefully chosen so that the axioms only place restrictions on relationships between the relevant groups.

With respect to the relevant groups, the results of Proposition 2.2 and Theorems 2.3, 2.4, and 2.5 would be entirely the same if the weakened forms of the axioms were used. The only difference would be with respect to the irrelevant groups — these axioms would not apply to them and therefore any non-degenerate rules would suffice.

3.3 Separating voters from issues

This paper has focused on the question of group identification, in which the binary opinions of n persons on n issues are simultaneously aggregated.[15] Alternatively, one might consider a more general model involving the simultaneous aggregation of the binary opinions of n persons on m issues, where $n \neq m$. All of the results in sections 2.3 and 2.4.2 are applicable to the more general case of simultaneous aggregation of binary opinions on multiple issues.

[15] Each of the n issues is the issue of whether a particular individual is a member of the group.

The special case where $m = 1$ corresponds a problem in the literature known as "judgment aggregation". In this case the set $\mathcal{G}$ corresponds to a set of logical propositions. For example, $a \in \mathcal{G}$ and $b \in \mathcal{G}$ might represent two elements of a crime, while $a \wedge b \in \mathcal{G}$ might represent the crime itself. A potentially desirable property is that it should not matter whether the court aggregates judgments about to the elements (a and b) or about the crime ($a \wedge b$). As is clear from Theorem 2.3, only agreement rules satisfy this property.[16] The first formal impossibility result in judgment aggregation was proved by List and Pettit (2002).

The judgment aggregation problem can be extended to the case where $m > 1$. For example, we might consider the case of two or more criminal co-defendants. The question of whether one defendant committed the first element of the crime is potentially related to the question of whether the other defendant committed that same element. The property discussed above implies that we must use an agreement rule. In some cases these rules may be plausible. If two defendants are being tried for conspiracy, for example, it might make sense to require that the jurors unanimously convict both to convict either.[17]

[16] When $m = 1$ agreement rules are also known as oligarchic rules.

[17] The idea here is that one cannot commit conspiracy without a co-conspirator. In general, American law has dealt with this problem by establishing rules which determine which opinions are to be aggregated. For example, the law might require that the court aggregate opinions about the crime and not about the elements. This is possible because all opinions are known. This solution is less plausible for the case of aggregating opinions about group membership because in that case it is not necessarily possible to know all opinions at the time they are aggregated.

Appendices

A Independence of the Axioms

Claim. *The meet separability, join separability, non-degeneracy, and anonymity axioms are independent.*

Proof. I present four rules. Each violates one axiom while satisfying the remaining three. This is sufficient to prove the claim.

Rule 1: Consider the rule f in which, for every $j \in N$, $f_j(P) = 1$ if and only if $P_{ij} = 1$ for some $i \in N$. This is a nomination rule but not a one-vote rule and therefore satisfies join separability and non-degeneracy but not meet separability (by Theorems 2.4 & 2.5).

Lastly, to show that it satisfies anonymity, let $j \in N$ and let π be a permutation of N. According to this rule, $f_j(P) = 1$ if and only if $P_{ij} = 1$ for some $i \in N$. Then $f_j(\pi P) = 1$ if and only if $(\pi P)_{ij} = P_{\pi(i)\pi(j)} = 1$ for some $i \in N$. Because this is true for any $i \in N$, $f_j(\pi P) = 1$ if and only if $P_{i\pi(j)} = 1$ for some $i \in N$. Furthermore, $\pi f_j(P) = f_{\pi(j)}(P) = 1$ if and only if $P_{i\pi(j)} = 1$ for some $i \in N$. Therefore, $\pi f_j(P) = f_j(\pi P)$ for all $j \in N$.

Rule 2: Consider the rule f in which, for every $j \in N$, $f_j(P) = 1$ if and only if $P_{ij} = 1$ for all $i \in N$. This is an agreement rule but not a one-vote rule and therefore satisfies meet separability and non-degeneracy but not join separability (by Theorems 2.3 & 2.5).

Lastly, to show that it satisfies anonymity, let $j \in N$ and let π be a permutation of N. According to this rule, $f_j(P, a) = 1$ if and only if $P_{ij} = 1$ for all $i \in N$. Then $f_j(\pi P) = 1$ if and only if $(\pi P)_{ij} = P_{\pi(i)\pi(j)} = 1$ for all $i \in N$. Because this must be true for all $i \in N$, $f_j(\pi P) = 1$ if and only if $P_{i\pi(j)} = 1$ for all $i \in N$. Furthermore, $\pi f_j(P) = f_{\pi(j)}(P) = 1$ if and only if $P_{i\pi(j)} = 1$ for all $i \in N$. Therefore, $\pi f_j(P) = f_j(\pi P)$ for all $j \in N$.

Rule 3: Let $a \in \mathcal{G}$ such that $a > b$ for no $b \in \mathcal{G}$. Consider the degenerate rule f in which $f(P, b) = \mathbf{1}$ if and only if $b \geq a$, and in which $f(P, b) = \mathbf{0}$, otherwise. This trivially satisfies the meet separability, join separability, and anonymity axioms, but violates non-degeneracy.

Rule 4: Consider the rule f in which, for every $j \in N$, $f_j(P) = 1$ if and only if $P_{1j} = 1$. This is a one-vote rule but is not self-identification and therefore satisfies the meet separability, join separability, and non-degeneracy axioms but violates anonymity. $\square$

B The generalized model

Formally, let $\mathcal{D}$ be a totally ordered set with a minimal element 0 and a maximal element 1. Let an aggregation rule be a mapping $f : \mathcal{D}^{N \times N} \times \mathcal{G} \to \mathcal{V}$.

One-opinion rules: For every individual j there exists (i, k) in $N \times N$ and $(d, r) \in \mathcal{D} \times \{0, 1\} \setminus \{(0, 1), (1, 0)\}$, such that, for all groups $a \in \mathcal{G}$, $f_j(P, a) = 1$ if $P_{ik} > d$ and $f_j(P, a) = 0$ if $P_{ik} < d$ and $f_j(P, a) = r$ if $P_{ik} = d$.

Note that in the case $\mathcal{D} = \{0, 1\}$ a one-opinion rule is equivalent to a one-vote rule.[18]

Theorem B.1. *A social rule f satisfies the meet separability, join separability, and non-degeneracy axioms if and only if it is a one-opinion rule.*

[18]In this case non-degeneracy implies that $r = d$.

Proof. That one-opinion rules satisfy the axioms is trivial. I show that any rule that satisfies the three axioms is a one-opinion rule. Let f satisfy the three axioms. By Proposition 2.2 f must satisfy group independence.

Let $P, Q \in \mathcal{V}^N$ and $j \in N$.

Define $\mathcal{P}_j \equiv \{P \in \mathcal{D}^{N\times N} : f_j(P) = 1\}$, $P^{j-} \equiv \wedge_{P\in\mathcal{P}_j} P$, and $P^{j+} \equiv \vee_{P\notin\mathcal{P}_j} P$.

Note that $f_j(P) = 1$ implies that $P \geq P^{j-}$ and that $f_j(P) = 0$ implies that $P \leq P^{j+}$. Therefore, for all $P \in \mathcal{D}^{N\times N}$, $P \geq P^{j-}$ and/or $P \leq P^{j+}$.

First, I establish that $P^{j-}_{ik} > 0$ for at most one pair $(i,k) \in N \times N$.

Suppose, contrariwise, that $\left|\{(i,k) \in N \times N : P^{j-}_{ik} > 0\}\right| > 1$. Without loss of generality, assume that $P^{j-}_{11} > 0$. Let $P^* \in \mathcal{V}^N$ such that all elements are 0 except that $P^*_{11} = 1$. Because $P^* \not\geq P^{j-}$ it follows that $f_j(P^*) = 0$. Let $P^\circ \in \mathcal{V}^N$ such that all elements are 1 except that $P^\circ_{11} = 0$. Because $P^\circ \not\geq P^{j-}$ it follows that $f_j(P^\circ) = 0$. Join separability implies that $f_j(P^* \vee P^\circ) = f_j(\mathbf{1}) = 0$. Meet separability implies that, for all $P \in \mathcal{D}^{N\times N}$, $f_j(P) = f_j(P \wedge \mathbf{1}) = f_j(P) \wedge f_j(\mathbf{1}) = 0$. The contradiction proves that $P^{j-}_{ik} > 0$ for at most one pair $(i,k) \in N \times N$. Similarly one can show that $P^{j+}_{ik} < 1$ for at most one pair $(i,k) \in N \times N$.

Second, I establish that $P^{j-}_{ik} > 0$ implies that $P^{j+}_{i'k'} = 1$ for all $(i',k') \neq (i,k)$.

Without loss of generality, assume that $(i,k) = (1,1)$. Suppose, contrariwise, that $P^{j-}_{11} > 0$ and that $P^{j+}_{12} < 1$. Let P° be as previously defined. We know that $P^\circ \not\geq P^{j-}$ and that $P^\circ \not\leq P^{j+}$. This contradiction proves that $P^{j-}_{ik} > 0$ implies that $P^{j+}_{i'k'} = 1$ for all $(i',k') \neq (i,k)$. Similarly one can show that $P^{j+}_{ik} < 1$ implies that $P^{j-}_{i'k'} = 0$ for all $(i',k') \neq (i,k)$.

Third, I establish that there exists (i,k) in $N\times N$ and $(d,r) \in \mathcal{D}\times\{0,1\}\setminus\{(0,1),(1,0)\}$, such that $f_j(P) = 1$ if $P_{ik} > d$ and $f_j(P) = 0$ if $P_{ik} < d$ and $f_j(P) = r$ if $P_{ik} = d$. The are four cases.

Case A: $P^{j-}_{ik} > 0$ for some $(i,k) \in N \times N$. First, it is clear that $P^{j-}_{ik} \geq P^{j+}_{ik}$; otherwise $P^{j+} >> P^{j-}$ which is a contradiction. Second, it is clear that there is no $x \in \mathcal{D}$ such that $P^{j-}_{ik} > x > P^{j+}_{ik}$. Otherwise, there is a profile P' such that $P'_{ik} = x$. But $P' \not\geq P^{j-}$ and $P' \not\leq P^{j+}$, and this is a contradiction. Then $P^{j-}_{ik} = d$ and $f_j(P^{j-}_{ik}) = r$. (Note that if $d = 1$ then $r = 1$ due to non-degeneracy.) Thus $f_j(P) = 1$ if $P_{ik} > d$, $f_j(P) = 0$ if $P_{ik} < d$, and $f_j(P) = r$ if $P_{ik} = d$.

Case B: $P^{j+}_{ik} < 1$ for some $(i,k) \in N \times N$. This is the dual of case A. Then $P^{j+}_{ik} = d$ and $f_j(P^{j+}_{ik}) = r$. (Note that if $d = 0$ then $r = 0$ due to non-degeneracy.) Thus $f_j(P) = 1$ if $P_{ik} > d$, $f_j(P) = 0$ if $P_{ik} < d$, and $f_j(P) = r$ if $P_{ik} = d$.

Case C: $P^{j-} = \mathbf{0}$. (This case is not possible if $\mathcal{D}$ is finite due to non-degeneracy.) There must be exactly one $(i,k) \in N \times N$ such that $P^{j+}_{ik} = 0$; otherwise $P^{j+} >> P^{j-}$, which would be a contradiction. Thus $d = r = 0$ and $f_j(P) = 1$ if $P_{ik} > d$ and $f_j(P) = r$ if $P_{ik} = d$.

Case D: $P^{j+} = \mathbf{1}$. This is the dual of case C. Thus, $d = r = 1$ and $f_j(P) = 0$ if $P_{ik} < d$ and $f_j(P) = r$ if $P_{ik} = d$.

□

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