Benchmarking

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Abstract

We introduce *benchmarking*, a method under which individuals are compared according to benchmarks, or important accomplishments. We show that benchmark rules are characterized by four axioms: transitivity, monotonicity, intersection dominance, and union dominance.

Introduction

Employers making hiring decisions commonly follow a two-stage process. The pile of candidates is first winnowed according to objective criteria. Difficult cases that remain are then resolved by an executive decision. While the latter choice may be made on the basis of gut feeling, the initial stage tends to follow a more formal process. This may be because the objective criteria make it easier to delegate the winnowing decision to administrative staff, or because an executive prefers to rely on more than mere instinct. We describe a path that this formal process may take; we call it *benchmarking*.

A *benchmark* is an important accomplishment. It can be a doctoral degree in economics, three years of work experience at the Securities and Exchange Commission, or a clerkship on the Supreme Court. Benchmarking is the process of comparing candidates according to benchmarks. A candidate is considered to be better than another if the former's résumé contains all of the latter's benchmarks. A common benchmark used in governmental hiring is the veterans' preference, according to which a military veteran is deemed superior to a non-veteran when they would otherwise be equivalent. Benchmarking may not provide a complete ranking; Alice may be a veteran without work experience, while Bob may be a non-veteran with work experience. But the benchmark rule nonetheless provides valuable comparisons that can be used by decision makers.

Benchmarking is used in a variety of settings. Universities may use benchmarks in admissions. Investment firms use benchmarks when deciding between projects. Consumers

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use benchmarks when buying computers. Governments frequently use benchmark rules in assigning priorities in procurement contracts.[1] Schools may be compared among among benchmarks, such as the scores that their students have received in math, science, history, and literature. While this practice is controversial, its' study is important; large sums of federal money are allocated to schools according to these metrics.[2]

Federal courts evaluating the decisions of administrative agencies often use benchmark rules in determining whether to uphold the agency decision. For example, the Federal Communications Commission was held to have acted unreasonably by awarding a television station to a broadcaster who was weaker than another on all relevant criteria used by the commission.[3] Administrative agencies themselves may choose whether particular criteria are important enough to qualify as benchmarks.[4]

Benchmark rules satisfy several nice properties. First, benchmarks enable us to order the candidates. The order is not necessarily complete; it may be that Alice and Bob are incomparable, but the order is *transitive*. If Alice is considered to be better than Chuck, and Chuck is better than Diane, then Alice will be better than Diane.

Second, benchmark rules are monotonic. Suppose that Alice has achieved every accomplishment that Bob has, and then some. Alice must be considered to be at least as good as Bob. But Bob may be her equal. What matters is not everything that Alice has done, but only the benchmarks — the accomplishments deemed important for the purposes of evaluation.

Third, benchmarks rules satisfy a property that we call *intersection dominance*. To understand this property, consider a third applicant, Chuck, whose accomplishments are exactly those common to both Alice and Bob. Suppose that, according to the rule, Alice is at least as good as Bob. In this case, intersection dominance requires that Chuck must also be at least as good as Bob. Why? Consider some accomplishment of Bob's that Chuck has not achieved. Alice must also not have achieved this accomplishment, yet it must have been immaterial in comparing Alice and Bob. Otherwise, Alice would not have been preferred to Bob. (Remember, we are not assuming completeness; Alice and Bob may be incomparable.) In essence, intersection dominance requires that if that accomplishment is irrelevant when comparing Bob with Alice, then that same accomplishment must also be irrelevant when comparing Bob with Chuck. Consequently, Chuck is at least as good as Bob. (Of course, if we add the monotonicity axiom, Bob will be at least as good as Chuck, and thus the two will be equivalent.)

Fourth, benchmarks rules satisfy a property that we call *union dominance*. To understand this property, consider a fourth applicant, Diane, who has achieved every accomplishment of Alice and every accomplishment of Bob, but not more. Suppose again that Alice is considered by the rule to be at least as good as Bob. Then union dominance

[1] For the criteria used in determining eligibility for federal contracts, see 48 C.F.R. Chapter 1.

[2] In particular, the "adequate yearly progress" requirements of the No Child Left Behind act (see 20 U.S.C. 6311) can function like a benchmark rule under which schools are compared to themselves in prior years. Over $14 billion was allocated for these grants in 2014.

[3] *Central Florida Enterprises, Inc. v. Federal Communications Commission*, 598 F.2d 37 (1979).

[4] For an example see *Baltimore Gas & Electric Co. v. Natural Resources Defense Council*, 462 U.S. 87 (1983), which upheld the decision of the Nuclear Regulatory Commission to ignore potential harm from the accidental release of spent nuclear fuel from long term storage.

requires the rule to consider Alice to be as good as Diane. The intuition is similar to that underlying intersection dominance. Consider some accomplishment of Diane that Alice has not achieved. Bob must have achieved it, yet it must have been immaterial in making a decision between Alice and Bob. Otherwise, Alice would not have been preferred to Bob. Union dominance requires that if that accomplishment is irrelevant when comparing Alice with Bob, then it must also be irrelevant when comparing Alice with Diane. Consequently, Alice is at least as good as Diane. (Again, the monotonicity axiom would imply that Diane is at least as good as Alice, so the two would be equivalent.)

Every benchmark rule satisfies transitivity, monotonicity, intersection dominance, and union dominance. But furthermore, the converse is true. Our main result is that a binary relation is a benchmark rule if and only if it satisfies these four axioms.

Benchmarking may be used when comparing scholars by their citation profiles, as in Chambers and Miller (2014b). Here, an accomplishment is a pair of two numbers (x, y) where the individual has x publications with at least y citations each. The step-based indices characterized by Chambers and Miller (2014b) are all benchmark rules, but the reverse is not true. For example, the h-index (Hirsch, 2005) and the i-10 index are two popular measures of scholarly accomplishment; each is a step-based index. However, many believe that multiple such indices should be used in practice. A method of comparison that determines Alice to be better than Bob if she is better according to both measures would not be a step-based index, but would be a benchmark rule. Benchmark rules are more versatile in that they can be applied to a wider array of problems than can be the step-based indices.

The relationship between the step-based indices and the benchmark rules is not a coincidence, but can be seen in the axioms as well. The intersection and union dominance axioms are weaker forms of the lattice-theoretic notions of meet and join homomorphisms used in Chambers and Miller (2014b). These properties were first studied in economics by Kreps (1979) and have since been studied in a wide variety of settings. For example, Hougaard and Keiding (1998), Christensen et al. (1999), and Chambers and Miller (2014a,b) study these axioms in the context of measurement, while Miller (2008), Chambers and Miller (2014a), Dimitrov et al. (2012), Leclerc (2011), and Leclerc and Monjardet (2013) study them in the context of aggregation.

Our work is also related to previous literature on incomplete preferences. It is a relatively easy corollary of our main result (proof available upon request) that $\succeq$ satisfies our axioms if and only if there is a family $\mathcal{R}$ of *complete* relations satisfying our axioms such that for all x, y, $x \succeq y$ if and only if for all $\succeq^* \in \mathcal{R}$, $x \succeq^* y$. Results of this type were pioneered by Dubra et al. (2004) for the expected-utility case. Other such results include Duggan (1999), for the case of general binary relations, Donaldson and Weymark (1998), Dushnik and Miller (1941), and Szpilrajn (1930).[5]

[5]See also the work of Ok (2002) on incomplete preferences in economic environments, and a choice theoretic foundation for incomplete preferences (Eliaz and Ok, 2006).

The Model

Let A be a set of accomplishments, and let $\leq$ be a partial order on A.[6] A subset $B \subseteq A$ is *comprehensive* if for all $b \in B$ and $a \in A$, $a \leq b$ implies that $a \in B$. Let X be the set of all finite comprehensive subsets of A. The set X forms a lattice when ordered by set inclusion. A **benchmarking rule** is a binary relation $\succeq$ on X for which there exists a set of benchmarks $B \subseteq A$ such that, for all $x, y \in X$, $x \succeq y$ if and only if $B \cap x \supseteq B \cap y$.

For a binary relation $\succeq$ on X, we define the following properties:

Transitivity: For all $x, y, z \in X$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.

Monotonicity: For all $x, y \in X$, if $x \supseteq y$, then $x \succeq y$.

Intersection dominance: for all $x, y \in X$, if $x \succeq y$, then $(x \cap y) \succeq y$.

Union dominance: For all $x, y \in X$, if $x \succeq y$, then $x \succeq (x \cup y)$.

The interpretation of these axioms is described in the introduction. We use these axioms to prove our main result.

Theorem 1. *A binary relation $\succeq$ on X satisfies transitivity, monotonicity, intersection dominance, and union dominance if and only if it is a benchmarking rule.*

The proof is given the appendix.

Examples

We provide three examples of sets that can be compared according to benchmark rules. These sets vary according to the specification of A and $\leq$, and of course, to the interpretation given to them.

Ranking Scholars

Academic institutions often use influence measures to compare scholars in terms of citations to their scientific publications. Popular influence measures include the h-index (Hirsch, 2005), the largest number h such that the scholar has at least h publications with at least h citations each, the $i10$-index, the number of publications with at least ten citations each, and the citation count, the combined number of citations to all of the author's publications.[7] Chambers and Miller (2014b) study a model of influence measures and characterize the family of step-based indices.

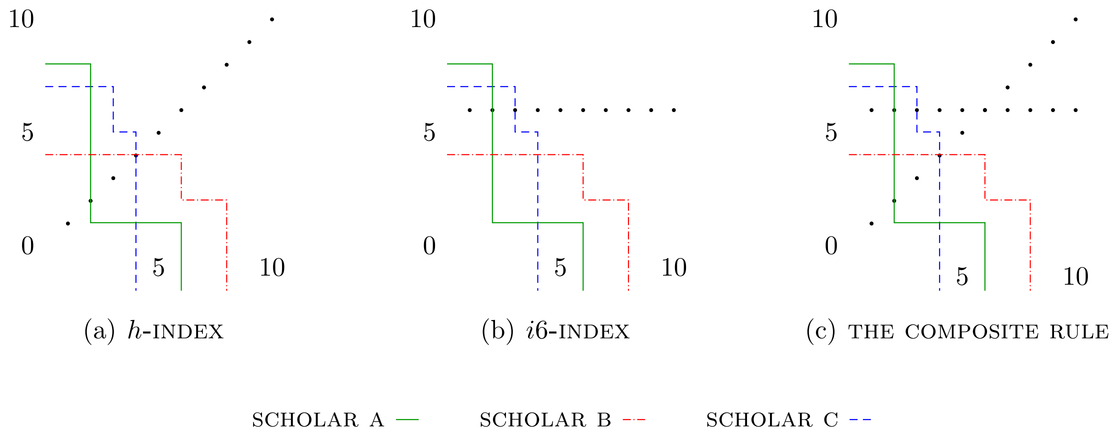
Influence measures can be studied in our framework. Let $A \equiv \mathbb{N} \times \mathbb{Z}_+$, the set of pairs of integers (m, n) where m is positive and n is non-negative, and let $\leq$ be the natural order, where $(m, n) \leq (m', n')$ if and only if $m \leq m'$ and $n \leq n'$. The set X of all finite comprehensive subsets of A is equivalent to the set of scholars. Benchmark

[6] A partial order is a binary relation which is *reflexive*, *transitive*, and *antisymmetric*.

[7] These particular measures are widely used, in part, due to their inclusion in the internet service "Google Scholar Profiles", available at http://scholar.google.com/.

rules are related to the step-based indices characterized in that paper: a relation $\succeq$ is a benchmark rule if and only if there is a collection $\{\succeq_i\}$ of step-based indices for which $x \succeq y$ if and only if $x \succeq_i y$ for all i. A benchmark rule is complete if and only if it can be represented by a single step-based index.

For example, the h-index and the $i6$-index (the number of publications with at least six citations each) are step-based indices and benchmark rules. The benchmarks for these rules are depicted in Figures 1(a) and 1(b) respectively. One can see that, according to the h-index, scholars B and C are equivalent and that each dominates scholar A. According to the $i6$-index, on the other hand, scholar C dominates scholar A, and each of them dominates scholar B. From these two indices we can construct a composite benchmark rule, under which a scholar is at least as good as another if the former scholar is at least as good according to both the h-index and the $i6$-index. This is shown in Figure 1(c). Under this composite rule, scholars A and B are incomparable, but both are dominated by scholar C. In the context of ranking scholars, every collection of step-based indices can generate a benchmark rule in this manner, and every benchmark rule can be generated by some collection of step-based indices.



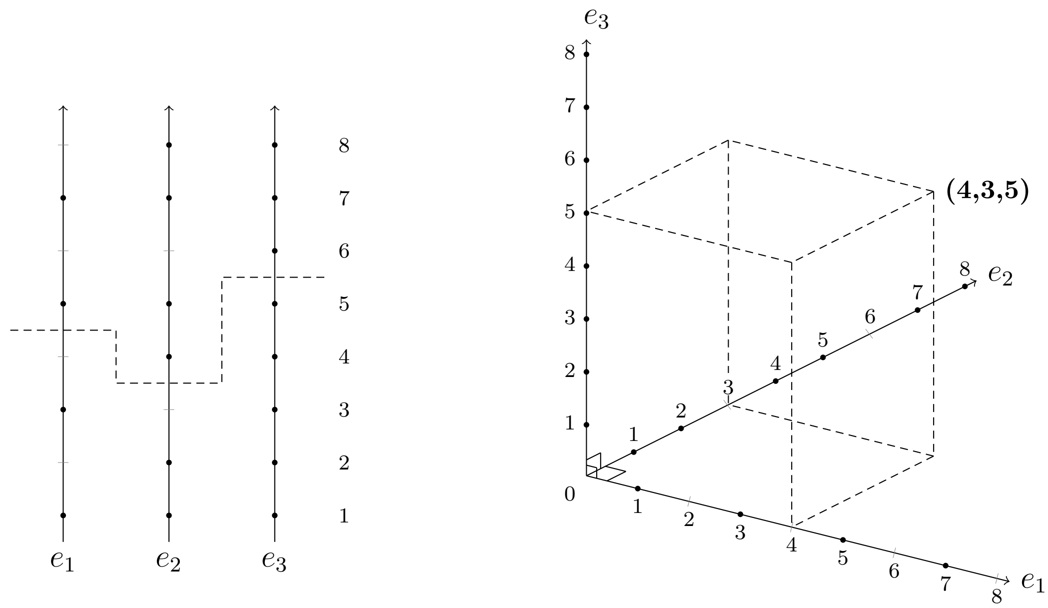
(a) h-INDEX (b) $i6$-INDEX (c) THE COMPOSITE RULE

Figure 1: Ranking Scholars.

Evaluating Schools

Under the No Child Left Behind act, schools are compared on the basis of their performance on a set of exams. Let E be a set of exams, and define $A \equiv E \times \mathbb{N}$, where an element (e, n) corresponds to a score of n on exam e. Define $\leq$ so that $(e, n) \leq (e', n')$ if and only if $e = e'$ and $n \leq n'$. A school is a finite comprehensive subset of A; that is, it consists of the schools' score on each exam, and all lesser scores. For example, if a school has received a scores of 2, 0, and 3 on exams e_1, e_2, and e_3, respectively, then the school is represented by the set $\{(e_1, 1), (e_1, 2), (e_3, 1), (e_3, 2), (e_3, 3)\}$. Note that in this example, the score of 2 on exam e_3 is imputed to the school because it received a higher score of 3.

Alternatively, we can represent each school as a vector in $s \in \mathbb{Z}_+^E$, where the s_e is the score on exam e. A benchmark is a score on a particular test; this is equivalent to a vector with exactly one non-zero element. For example, the vector $(0, 4, 0)$ denotes the benchmark of having achieved a score of 4 on the 2nd exam. A school s achieves benchmark b if $b_e \leq s_e$ for all $e \in E$. There is a bijection from the set X of all finite comprehensive subsets of A to the set of vectors $\mathbb{Z}_+^E$. To see this, let $x \in X$. Then there is a vector $s^x \in \mathbb{Z}_+^E$, where $s_e^x = \max\{0, n : (e, n) \in x\}$. Conversely, if we begin with $s \in \mathbb{Z}_+^E$, we can define the set $x^s \equiv \{(e, n) : n \leq s_e\}$.



(a) Benchmark Representation (b) Vector Representation

Figure 2: Ranking Schools.

Figure 2 depicts both representations of schools. In the benchmark representation (Figure 2(a)), each exam is represented by a separate vertical line containing the possible scores on that exam. A school takes the form of a finite comprehensive subset, which implies that the school can be represented by its' upper boundary. In the example, the school has achieved scores of 4, 3, and 5 on the respective exams. In the vector representation (Figure 2(b)), on the other hand, the same school is depicted by the vector $(4, 3, 5)$. In both representations, benchmarks are represented by black dots on the axes. One can see that the highest benchmarks this school has achieved are scores of three on the first exam, two on the second, and five on the third.

Of course, this representation of schools can be applied to any case in which alternatives are compared among multiple dimensions. Academics, for example, may be compared according to their accomplishments in different areas, such as the numbers of papers published, citations received, seminars given, and conferences attended. In this framework, the citation count can satisfy the axioms, although it would not satisfy the axioms in the case of ranking scholars, above. The dimensions may also represent criteria used in multicriterial decision-making.[8]

Unordered Accomplishments

We can also compare all finite subsets of any set A, regardless of comprehensiveness. To do this, we define $\leq$ such that no two distinct elements are comparable (*i.e.*, so that $a \leq b$ implies that $a = b$). A set of benchmarks is pictured in Figure 3. Here, scholars A and B are incomparable, because each contains benchmarks not contained by the other. Similarly, scholars A and C are also not comparable. However, one can see that scholar B dominates scholar C.

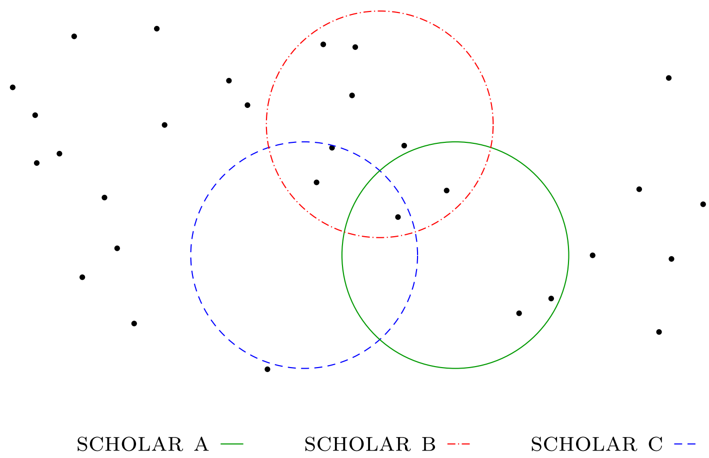


Figure 3: Unordered Accomplishments.

Conclusion

We have described a method of comparison that we term "benchmarking." Benchmark rules are characterized by four axioms, transitivity, monotonicity, intersection dominance, and union dominance.

Benchmark rules are not necessarily complete, and can be used in cases where completeness is considered undesirable or is otherwise not required. One can see that a benchmark rule will satisfy completeness if and only if the benchmarks are totally ordered. In the case of ranking scholars, this implies that they must form a step-based index. In the case of evaluating schools, it implies that only a single test will matter. In the unordered case, it implies that there can be only one benchmark.

In some cases it may seem as if, in practice, the benchmark rule is simply a comparison according to set-inclusion. This does not necessarily mean that all potential accomplishments are benchmarks. Alternatively, it may be that only benchmarks are included on résumés. This would be expected if the rule were to be known with certainty.

[8]See Spitzer (1979) and Katz (2011) for applications of multicriterial decision-making in law.

In other cases, however, résumés often include accomplishments that are not benchmarks. One might ask why this would occur. As we noted, this might come across as a result of uncertainty about the rule. However, there are other possibilities. One is that the applicant finds it efficient to use the same résumé for multiple employers, or in multiple markets, where different rules may be applied. Alternatively, as we explained in the introduction, the benchmark rule might be used only as a first step in sorting applicants; the otherwise extraneous information might still be relevant when making an executive decision.

Appendix

The proof of Theorem 1 makes use of the concept of an *interior operator*,[9] a function $i : X \to X$ for which the following three properties are satisfied:

Contractivity: for all $x \in X$, $i(x) \subseteq x$.

Monotonicity: for all $x, y \in X$, $x \subseteq y$ implies $i(x) \subseteq i(y)$.

Idempotence: for all $x \in X$, $i(i(x)) = i(x)$.

Proof of Theorem 1. **If:** Let $\succeq$ be a benchmarking rule and with benchmarks B. The transitivity of $\succeq$ follows from the transitivity of $\supseteq$. To see this, let $x \succeq y$ and $y \succeq z$. It follows that $B \cap x \supseteq B \cap y$ and $B \cap y \supseteq B \cap z$. Hence $B \cap x \supseteq B \cap z$ and therefore $x \succeq z$. To see that $\succeq$ is monotonic, note that $x \supseteq y$ implies that $B \cap x \supseteq B \cap y$ and therefore that $x \succeq y$. Lastly, let $x \succeq y$. Then $B \cap x \supseteq B \cap y$. Intersection dominance follows from the fact that $B \cap (x \cap y) = (B \cap x) \cap (B \cap y) = B \cap y$ and therefore $x \cap y \succeq y$. Union dominance follows from the fact that $B \cap x = (B \cap x) \cup (B \cap y) = B \cap (x \cup y)$ and hence $x \succeq x \cup y$.

Only if: Let $\succeq$ satisfy the four axioms transitivity, monotonicity, intersection dominance, and union dominance. Define, for every $x \in X$, the function

$$i(x) \equiv \bigcap \{z \subseteq x : z \succeq x\}.$$

Notice that because $|\{z \in X : z \subseteq x\}| < +\infty$ for all x, it follows that $0 < |\{z \subseteq x : z \succeq x\}| < +\infty$, so $i(x) \in X$. Observe also that $i(x) = \bigcap \{z : z \succeq x\}$, since for every z such that $z \succeq x$, we know that $x \cap z \succeq x$ (by intersection dominance) and that $x \cap z \subseteq x$. We claim that $i(\cdot)$ is an interior operator satisfying, for all $x, y \in X$, (1) $x \succeq y$ if and only if $i(x) \supseteq i(y)$ and (2) $i(x) \cup i(y) = i(x \cup y)$.

First, the fact that $x \in \{z : z \succeq x\}$ implies that $i(x) = \bigcap \{z : z \succeq x\} \subseteq x$, so $i(\cdot)$ is contractive. Next, we claim that $x \sim i(x)$. To see this, by contractivity and monotonicity

[9] An interior operator is a dual concept to that of a closure operator (Ward, 1942). These objects are intimately connected with the theory of Galois connections and residuated mappings; see, e.g. Blyth and Janowitz (1972). Galois connections have recently found economic application in Nöldeke and Samuelson (2015).

of $\succeq$ we know that $x \succeq i(x)$. Furthermore, by intersection dominance, $z \succeq x$ implies $x \cap z \succeq x$, and thus by finite induction it follows that $i(x) = \bigcap\{z \subseteq x : z \succeq x\} \succeq x$.

We prove that $x \succeq y$ if only if $i(x) \supseteq i(y)$. First, let $x \succeq y$. It follows that $\{z : z \succeq y\} \supseteq \{z : z \succeq x\}$ and therefore that $i(x) \supseteq i(y)$. To show the converse, let $i(x) \supseteq i(y)$. By monotonicity of $\succeq$ it follows that $x \sim i(x) \succeq i(y) \sim y$, and thus by transitivity, $x \succeq y$.

To see that $i(\cdot)$ satisfies monotonicity, let $x \supseteq y$. By monotonicity of $\succeq$ it follows that $x \succeq y$, and therefore that $i(x) \supseteq i(y)$. To see that $i(\cdot)$ satisfies idempotence, note that because $x \sim i(x)$ it follows that the sets $\{z : z \succeq z\}$ and $\{z : z \succeq i(x)\}$ coincide.

Finally, let $x, y \in X$. Recall that $x \sim i(x)$ and $y \sim i(y)$. By transitivity and monotonicity of $\succeq$, we know that $i(x) \cup i(y) \succeq x$ and $i(x) \cup i(y) \succeq y$. By union dominance and monotonicity of $\succeq$, $y \cup i(x) \cup i(y) \succeq i(x) \cup i(y)$. By transitivity and monotonicity of $\succeq$, we know that $x \succeq y \cup i(x) \cup i(y)$, thus by union dominance and monotonicity of $\succeq$, $x \cup y \cup i(x) \cup i(y) \succeq y \cup i(x) \cup i(y)$. By transitivity, $x \cup y \cup i(x) \cup i(y) \succeq i(x) \cup i(y)$. By contractivity, $x \cup y \cup i(x) \cup i(y) = x \cup y$, hence $x \cup y \succeq i(x) \cup i(y)$. Using the fact that for arbitrary a, $i(a) = \bigcap\{z : z \succeq a\}$, it follows that $i(x \cup y) \subseteq i(x) \cup i(y)$. Lastly $i(x) \subseteq i(x \cup y)$ and $i(y) \subseteq i(x \cup y)$ (from the monotonicity of c), and thus $i(x) \cup i(y) \subseteq i(x \cup y)$. Therefore $i(x) \cup i(y) = i(x \cup y)$.

For a subset $C \subseteq A$, we define $\mathcal{K}(C)$ as its *comprehensive hull*; the smallest comprehensive set containing C. To simplify notation, for an element $a \in A$, we write $\mathcal{K}(a)$ in place of $\mathcal{K}(\{a\})$.

Define $B \equiv \{a \in A : \mathcal{K}(a) = i(\mathcal{K}(a))\}$. We claim that $i(x) = \mathcal{K}(B \cap x)$. To see this, let $a \in \mathcal{K}(B \cap x)$. It follows that there exists $b \in B \cap x$ for which $a \leq b$. Because $b \in x$ it follows that $\mathcal{K}(b) \subseteq x$, and by monotonicity of $i(\cdot)$, $i(\mathcal{K}(b)) \subseteq i(x)$. Because $b \in B$ it follows that $\mathcal{K}(b) = i(\mathcal{K}(b))$ and therefore $b \in i(x)$. Because $i(x)$ is comprehensive and $a \leq b$ it follows that $a \in i(x)$. Conversely, let $a \in i(x)$. Let $b \in i(x)$ such that $a \leq b$ and b on in the comprehensive frontier of $i(x)$, so that $i(x) \setminus \{b\} \in X$. Then $i(x) = \mathcal{K}(b) \cup \mathcal{K}(i(x) \setminus \{b\})$. By idempotence and the join homomorphism property, $i(x) = i(i(x)) = i(\mathcal{K}(b)) \cup i(i(x) \setminus \{b\})$. Because $b \in i(x)$ and $b \notin i(x) \setminus \{b\}$ it follows that $b \in i(\mathcal{K}(b))$ and hence $\mathcal{K}(b) \subseteq i(\mathcal{K}(b))$. By contractivity it follows that $\mathcal{K}(b) = i(\mathcal{K}(b))$ and hence $b \in B$. Because $b \in i(x)$ it follows from contractivity that $b \in x$ and therefore that $b \in B \cap x$. Because $a \leq b$ it follows that $a \in \mathcal{K}(B \cap x)$ and hence $i(x) = \mathcal{K}(B \cap x)$.

It follows then that $x \succeq y$ if and only if $\mathcal{K}(B \cap x) \supseteq \mathcal{K}(B \cap y)$. It remains to be shown that $\mathcal{K}(B \cap x) \supseteq \mathcal{K}(B \cap y)$ if and only if $B \cap x \supseteq B \cap y$. One direction (if) is trivial. To see the converse, let $\mathcal{K}(B \cap x) \supseteq \mathcal{K}(B \cap y)$ and let $a \in B \cap y$. Then $a \in \mathcal{K}(B \cap x)$ and thus there exists $b \in B \cap x$ such that $a \leq b$. By construction $a \in B$, and $b \in x$, and $a \leq b$ implies that $a \in x$ and hence $a \in B \cap x$.

Independence of the Axioms: We describe four rules: each satisfies three axioms while violating the fourth.

Transitivity: Let $\succeq_1, \succeq_2, \succeq_3$ be benchmark rules such that there exist $x, y, z \in X$ such that $\{i : x \succeq_i y\} = \{1, 2\}$, $\{i : y \succeq_i z\} = \{2, 3\}$, and $\{i : x \succeq_i z\} = \{2\}$. Let $\succeq$ be the rule where $x \succeq y$ iff $|\{i : x \succeq_i y\}| \geq 2$. To see that this rule is monotonic, let $x \supseteq y$. Because the benchmark rules are monotone, it follows that $x \succeq_i y$ for all i and thus $x \succeq y$. To see that the rule is intersection dominant, let $x \succeq y$. It follows that $|\{i : x \succeq_i y\}| \geq 2$. Because the benchmark rules are intersection dominant, $x \succeq_i y$ implies that $x \cap y \succeq_i y$.

Therefore $|\{i : x \cap y \succeq_i y\}| \geq 2$ implies that $x \cap y \succeq y$. To see that the rule is union dominant, let $x \succeq y$. It follows that $|\{i : x \succeq_i y\}| \geq 2$. Because the benchmark rules are union dominant, $x \succeq_i y$ implies that $x \succeq_i x \cup y$. Therefore $|\{i : x \succeq_i x \cup y\}| \geq 2$ implies that $x \succeq x \cup y$. To see that the rule violates transitivity, let $x, y, z \in X$ such that $\{i : x \succeq_i y\} = \{1, 2\}$, $\{i : y \succeq_i z\} = \{2, 3\}$, and $\{i : x \succeq_i z\} = \{2\}$. Then $x \succeq y$ and $y \succeq z$ but not $x \succeq z$.

Monotonicity: Let $\succeq$ be the rule where $x \succeq y$ if and only if $x = y$. Transitivity of $\succeq$ follows from transitivity of $=$. To show that $\succeq$ is intersection dominant, let $x \succeq y$. Because $s = y$ it follows that $x \cap y = y$, hence $x \cap y \succeq y$. To show that $\succeq$ is union dominant, let $x \succeq y$. Because $x = y$ it follows that $x \cup y = x$, hence $x \succeq x \cup y$. To show that $\succeq$ is not monotone, let $x \subsetneq y$. Then $y \not\succeq x$, a violation of monotonicity.

Intersection Dominance: Let $B \equiv \{\{1\}, \{1, 3\}, A\}$, and let $\succeq$ be the rule where $x \succeq y$ if and only if $\{b \in B : x \subseteq b\} \subseteq \{b \in B : y \subseteq b\}$. To show that $\succeq$ is transitive, let $x, y, z \in X$ such that $x \succeq y \succeq z$. Then $\{b \in B : x \subseteq b\} \subseteq \{b \in B : y \subseteq b\} \subseteq \{b \in B : z \subseteq b\}$, and consequently $x \succeq z$. To see that $\succeq$ is monotonic, let $x \supseteq y$, and note that for all $b \in B$, $x \subseteq b$ implies that $y \subseteq b$. This implies that $\{b \in B : x \subseteq b\} \subseteq \{b \in B : y \subseteq b\}$ and therefore $x \succeq y$. To show that $\succeq$ is union dominant, let $x, y \in X$ such that $x \succeq y$. This implies that $\{b \in B : x \subseteq b\} \subseteq \{b \in B : y \subseteq b\}$. For all $b \in B$, $x \cup y \subseteq b$ if and only if $x \subseteq b$ and $y \subseteq b$, hence $\{b \in B : x \subseteq b\} \subseteq \{b \in B : x \cup y \subseteq b\}$. Thus, $x \succeq x \cup y$. To show that $\succeq$ fails intersection dominance, note that $\{1, 2\} \succeq \{1, 3\}$, because $\{b \in B : \{1, 2\} \subseteq b\} = \{A\} \subseteq \{A, \{1, 3\}\} = \{b \in B : \{1, 3\} \subseteq b\}$. However, $\{b \in B : \{1, 2\} \cap \{1, 3\} \subseteq b\} = B \not\subseteq \{A, \{1, 3\}\} = \{b \in B : \{1, 3\} \subseteq b\}$ and therefore $\{1, 2\} \cap \{1, 3\} \not\succeq \{1, 3\}$.

Union Dominance: Let $B \equiv \{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$, and let $\succeq$ be the associated benchmark rule. To show that $\succeq$ is transitive, let $x, y, z \in X$ such that $x \succeq y \succeq z$. Then $\{b \in B : b \subseteq x\} \supseteq \{b \in B : b \subseteq y\} \supseteq \{b \in B : b \subseteq z\}$, and consequently $x \succeq z$. To see that $\succeq$ is monotonic, let $x \supseteq y$, and note that for all $b \in B$, $b \subseteq y$ implies that $b \subseteq x$. This implies that $\{b \in B : b \subseteq x\} \supseteq \{b \in B : b \subseteq y\}$ and therefore $x \succeq y$. To show that $\succeq$ is intersection dominant, let $x, y \in X$ such that $x \succeq y$. This implies that $\{b \in B : b \subseteq x\} \supseteq \{b \in B : b \subseteq y\}$. For all $b \in B$, $b \subseteq x \cap y$ if and only if $b \subseteq x$ and $b \subseteq y$, hence $\{b \in B : b \subseteq x \cap y\} \subseteq \{b \in B : x \cup b \subseteq y\}$. Thus, $x \cap y \succeq y$. To show that $\succeq$ fails union dominance, note that $\{1, 2\} \succeq \{2, 3\}$, because $\{b \in B : b \subseteq \{1, 2\}\} = \{\{1\}, \{1, 2\}\} \supseteq \varnothing = \{b \in B : b \subseteq \{2, 3\}\}$. However, $\{b \in B : b \subseteq \{1, 2\}\} = \{\{1\}, \{1, 2\}\} \not\supseteq B = \{b \in B : b \subseteq \{1, 2\} \cup \{2, 3\}\}$ and therefore $\{1, 2\} \not\succeq \{1, 2\} \cup \{2, 3\}$. □

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