Arrow's Theorem Without Transitivity

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Abstract

In light of research indicating that individual behavior may violate standard rationality assumptions, we introduce a model of preference aggregation in which neither individual nor collective preferences must satisfy transitivity or other coherence conditions. We introduce an ordinal rationality measure to compare preference relations. Using this measure, we introduce a new axiom, monotonicity, which requires the collective preference to become more rational when the individual preferences become more rational. We show that no collective choice rule satisfies monotonicity and the standard Arrovian assumptions: unrestricted domain, weak Pareto, independence of irrelevant alternatives, and nondictatorship.

JEL Codes: D60; D70; D71.

1 Introduction

The core model of preference aggregation introduced by Arrow (1963) contains a strong assumption about the rationality of preferences. In particular, individual preferences are assumed to be reflexive, complete, and transitive. The first two properties are often considered richness conditions, while transitivity is a coherence condition.[1]

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[1] By coherence, we refer to the class of formal conditions on preference relations that includes transitivity, quasi-transitivity, acyclicity, semi-transitivity, and the interval order (see Bossert and Suzumura, 2007).

In light of behavioral research casting doubt on the assumption of transitivity (Tversky, 1969), we modify Arrow's model to remove the requirement that individual preference relations be transitive or satisfy other known coherence conditions.[2]

Having removed this rationality requirement, we use this new framework to study an important question about collective preferences. Do more rational individuals create a more rational society?[3]

In other words, suppose that we have a set of agents, at least one of whom is less than fully rational, with a not-necessarily-rational collective preference. We may be able to induce an agent to "correct" his preferences by pointing out that his behavior violates minimal conditions of rationality. When the agent becomes more rational as a result of this correction, will the collective preference become more rational as a result?

Our conclusion is negative; if group decisions are made in a non-dictatorial way, it is possible that an increase in individual rationality may lead to a decrease in collective rationality. It may be possible to manipulate a group by helping individuals correct their mistakes. On a more practical note, a group of people may become more susceptible to "Dutch books" when the individuals' susceptibility lessens.

We illustrate this problem by means of a simple example. It is clear that many collective choice rules satisfy the remaining assumptions imposed by Arrow (1963): unrestricted domain, weak Pareto, independence of irrelevant alternatives, and non-dictatorship. A simple example is the method of majority decision, in which alternative x is weakly preferred to alternative y whenever the majority weakly prefers x to y. (For more on the method of majority decision, see Sen (1964, 1966).)

However, the method of majority decision has an undesirable property. Suppose that there are three individuals, Alice, who prefers x to y, y to z, and x to z, Bob, who prefers z to x, x to y, and z to y, and Carol who prefers x to z, z to y, and y to x. Alice and Bob have transitive preferences, but Carol does not. By the method of majority decision, x is preferred to y, z is preferred to y, and x is preferred to z, leading to a transitive and rational collective preference. However, suppose that Carol realizes that her preferences are irrational and seeks to "correct" them. She decides to retain her view that y is preferred to x but changes her opinion of z, so that she now prefers y to z and z to x. As a consequence, the method of majority decision leads to the collective preference x to y, y to z, and z to x, and is no longer transitive. In this case, the collective preference became less rational *because* Carol became more rational.

[2]With a similar motivation, de Clippel (2012) studies mechanism design without the assumption that individuals make choices as if they are maximizing a preference relation.

[3]We identify society with a *preference-aggregation method* that satisfies the classic Arrovian restrictions. We recognize, of course, that this general question may be studied in a different setting, and that even those following the aggregation approach may address the question without the imposition of Arrow's conditions. We do not attempt to provide the "ultimate answer" to this question. Rather, we provide one meaningful answer within one of the most basic frameworks in economics.

We show that this problem is not unique to method of majority decision. In fact, every collective choice rule which satisfies the remaining assumptions of Arrow (1963) will have this undesirable property. For every such method it is possible that an increase in the rationality of individual preferences will lead to a decrease in the rationality of the collective preference.[4]

Our formal model can be described as follows. First, we study a modified version of collective choice rules in which neither the individual nor collective preferences are required to be rational.[5] We assume only that preference relations be reflexive and complete. Thus each individual's preferences can be described by a reflexive and complete relation, and the collective preference can be described by a reflexive and complete relation as well. We implicitly assume that every possible combination of individual preference relations is possible; i.e. that Arrow's unrestricted domain axiom holds in this setting.

Using accepted notions of rationality, we introduce the concept of an ordinal rationality measure, and identify some minimal conditions that any reasonable rationality measure should satisfy.[6] We formulate an axiom, *monotonicity*, to address the problem exhibited by majority rule in the above example.[7] This axiom requires that the collective choice rule be monotonic with respect to the rationality measure; that is, if individual preferences change and become more rational, then the collective preference should become more rational, if it changes at all.

In addition to monotonicity, we impose the three additional axioms of Arrow (1963): *weak Pareto*, which requires that society strictly prefer x to y whenever every individual strictly prefers x to y, *independence of irrelevant alternatives*, which requires that a change in the opinions about alternative z does not affect the relative ranking of alternatives x and y, and *nondictatorship*, which requires that no individual be a dictator. We show that the four axioms are incompatible. In other words, regardless of which ordinal rationality measure we choose, we cannot find a collective choice rule which is monotonic, weakly Paretian, independent of irrelevant alternatives, and nondictatorial.

Previous studies have sought to weaken the assumption of rationality in Arrow (1963) by permitting a wider range of collective preferences. The case of quasi-

[4]For this reason, we do not argue that the method of majority decision is any worse that any other method in this context. Several studies, including Sen (1966), Inada (1969), and Batra and Pattanaik (1972), examine the conditions under which pairwise majority does not lead to cycles. Dasgupta and Maskin (2008) provide an argument that the method of majority decision is more robust than other voting methods in that it violates the standard axioms on fewer domains.

[5]While we study the aggregation of preference relations, all of our results would hold if the paper was formulated with respect to choice functions.

[6]Our conditions are satisfied by all known measures of rationality such as the measures of Varian, Afriat, Houtman-Maks, as well as the money pump index and the minimal swaps index (see Echenique et al., 2011; Apesteguia and Ballester, 2013).

[7]Our "monotonicity" axiom is not related to the traditional monotonicity axiom ("positive association of individual and social values") studied in social choice (see Arrow, 1963).

transitive collective preferences was studied by many including Gibbard (1969), Sen (1969, 1970), Schick (1969), and Mas-Colell and Sonnenschein (1972), and that of acyclic preferences by Mas-Colell and Sonnenschein (1972) and Blair and Pollak (1982). For more see Sen (1977).[8]

Other scholars have tried to avoid the negative conclusions of Arrow (1963) by moving in the opposite direction. Instead of expanding the range of admissible collective preferences, these studies restrict the domain of allowable preferences. The most prominent example is that of the single-peaked preference restriction of Black (1948a,b) and Arrow (1963).

The most closely related literature is the study of tournaments, which are described by binary relations which are antisymmetric and complete. Unlike the preference relations we study, tournaments do now allow for the possibility of ties. In this context, Monjardet (1978) shows that a collective choice rule that (a) maps every profile of transitive preferences into a transitive preference, (b) satisfies the independence of irrelevant alternatives axiom and (c) satisfies a non-imposition axiom is either dictatorial or "persecutive." Roughly speaking, persecutive means that the decisive coalitions are all the coalitions that do not contain a certain individual i. A related result can be found in Barthelemy (1982). As far as we can tell, the monotonicity axiom that we present is new to this paper.

2 Model and result

Let X be a set of alternatives, $|X| \geq 3$. A preference relation R on X is (a) **complete** if for all $x, y \in X$, $x \neq y$ implies that either xRy or yRx, (b) **reflexive** if for all $x \in X$, xRx, and (c) **transitive** if for all $x, y, z \in X$, xRy and yRz implies that xRz. Let $\mathscr{R}$ be the set of all complete and reflexive preference relations on X. A **preference ordering** is a preference relation which is complete, reflexive, and transitive. Let $\mathscr{R}^* \subseteq \mathscr{R}$ be the set of preference orderings on X.

For a preference relation $R \in \mathscr{R}$ we denote by P its asymmetric component; that is, xPy if xRy but not yRx. A preference relation is **acyclic** if, for every $k \geq 3$ and every $x^1, ..., x^k \in X$, $x^i P x^{i+1}$ for all $i < k$ implies that $x^k P x^1$ does not hold. Let $\mathscr{R}^a \subseteq \mathscr{R}$ be the set of preference relations which are complete, reflexive, and acyclic. It is well known that $\mathscr{R}^* \subsetneq \mathscr{R}^a \subsetneq \mathscr{R}$ (see Suzumura, 1983).

For $Y \subseteq X$, denote by $\mathscr{R}|_Y$ the set all complete and reflexive preference relations on Y, and denote by $\mathscr{R}^*|_Y$ the set all preference orderings in $\mathscr{R}|_Y$. For $R \in \mathscr{R}$ and $Y \subseteq X$, denote by $R|_Y \in \mathscr{R}|_Y$ the restriction of R to Y.

Let $N \equiv \{1, ..., n\}$ be a finite set of agents, $n \geq 2$. A *profile* $\mathbf{R} = (R_1, \cdots, R_n) \in \mathscr{R}^N$ is a vector of preference relations, one for each agent. A **collective choice rule**

[8]A related literature looks at the case in which preferences are transitive but not necessarily complete (see Baucells and Shapley, 2008; Pini et al., 2009).

is a mapping $f\colon \mathscr{R}^N \to \mathscr{R}$.[9] We define $R_0 \equiv f(\mathbf{R})$ to be the social relation, and we denote by P_0 its asymmetric component. A set of elements $Y \subseteq X$ is **top-ranked** in profile $\mathbf{R}$ if $a \in Y$, $b \in X \setminus Y$, and $i \in N$ implies that aP_ib.

A **rationality measure** is a binary relation $\succcurlyeq$ on $\mathscr{R}$ which satisfies the following properties:

1. For all $R \in \mathscr{R}$, $R \succcurlyeq R$.
2. For all $R' \in \mathscr{R}^*$ and $R \in \mathscr{R}$, $R \succcurlyeq R'$ implies that $R \in \mathscr{R}^*$.
3. For all $R^* \in \mathscr{R}^*$ and $R \in \mathscr{R} \backslash \mathscr{R}^a$: if there is a three-element set $Y \subseteq X$ which is top-ranked in both relations, such that $R^* \big|_{X\backslash Y} = R \big|_{X\backslash Y}$, then $R^* \succcurlyeq R$.

For two profiles $\mathbf{R}, \mathbf{R}' \in \mathscr{R}^N$ we write $\mathbf{R} \succcurlyeq \mathbf{R}'$ if $R_i \succcurlyeq R'_i$ for all $i \in N$.

Property 1, known as reflexivity, requires each preference relation to be "at least as rational" as itself. Property 2 requires that only a transitive preference ordering can be at least as rational as another transitive preference ordering. Property 3 requires that every transitive preference ordering must be at least as rational as every cyclic preference relation, provided that the two relations are identical except for the three top-ranked elements.

A wide range of rationality measures satisfies these conditions. We provide two examples. The simplest rationality measure $\succcurlyeq'$ is one for which $R^* \succcurlyeq' R$ if and only if $R^* \in \mathscr{R}^*$ and $R \in \mathscr{R} \setminus \mathscr{R}^*$. A more complicated rationality measure can incorporate the structure of coherence properties studied in the social choice literature. For example, we can define a rationality measure $\succcurlyeq''$ such that $R^* \succcurlyeq'' R$ if and only if there exists an $\mathscr{C} \in \{\mathscr{R}^*, \mathscr{R}^a\}$ such that $R^* \in \mathscr{C}$ but $R \notin \mathscr{C}$.[10]

Our first axiom, monotonicity, requires that if preference relations change, and each individual's new preference relation stays at least as rational as it was before the change, then the social preference must stay at least as rational.

Monotonicity: For all $\mathbf{R}, \mathbf{R}' \in \mathscr{R}^N$, if $\mathbf{R} \succcurlyeq \mathbf{R}'$ then $R_0 \succcurlyeq R'_0$.

The following three axioms were introduced by Arrow (1963); for brevity, we will not discuss them.

Weak Pareto: For every $\mathbf{R} \in \mathscr{R}^N$ and $x, y \in X$, if xP_iy for all $i \in N$, then xP_0y.

Independence of Irrelevant Alternatives: For all $Y \subseteq X$ and $\mathbf{R}, \mathbf{R}' \in \mathscr{R}^N$, if $\mathbf{R}\,|_Y = \mathbf{R}'\,|_Y$, then $R_0\,|_Y = R'_0\,|_Y$

[9] This is a slight change from the standard definition, in which the domain of a collective choice rule is a set of preference orderings (see Sen, 1970).

[10] The three classes were chosen for the ease of the exposition. Clearly a rationality measure can incorporate any number of classes, and these not be totally ordered through set inclusion. In particular, the rationality measure can incorporate the coherence properties of quasi-transitivity, semi-transitivity and the interval order. See Cato (2012).

An individual $d \in N$ is a *dictator* if, for all $\mathbf{R} \in \mathscr{R}^N$, xP_dy implies that xP_0y.

Non-Dictatorship: There does not exist a dictator.

We can now turn to the main result.

Theorem 1. *There does not exist a collective choice rule that satisfies monotonicity, weak Pareto, independence of irrelevant alternatives, and non-dictatorship. Furthermore, the axioms are independent.*

To prove this theorem we make use of the following lemma. For a coalition $K \subseteq N$, we define $x\bar{D}_Ky$ as the statement that the coalition K is decisive for x over y; that is, if xP_iy for all $i \in K$, then xP_0y. Similarly, we define xD_Ky as the statement that the coalition K is decisive for x over y when all others are opposed; that is, if xP_iy for all $i \in K$ and yP_ix for all $i \notin K$, then xP_0y.

Lemma 1. *If a collective choice rule f satisfies monotonicity, weak Pareto, and independence of irrelevant alternatives, then whenever xD_Ky for a coalition $K \subseteq N$ and some pair of alternatives $x, y \in X$, it follows that $w\bar{D}_Kz$ for every pair $w, z \in X$.*

Proof of Lemma 1. Let the collective choice rule f satisfy the monotonicity, weak Pareto, and independence of irrelevant alternatives axioms. Let $K \subseteq N$ and $x, y \in X$ such that xD_Ky.

Step one. We claim that, for all $z \in X \setminus \{x, y\}$, if $\mathbf{R} \in \mathscr{R}^N$ such that (a) $R_i\big|_{\{x,y,z\}} \in \mathscr{R}^*\big|_{\{x,y,z\}}$ for all $i \in N$, (b) xP_iy for all $i \in K$, and (c) $R_i\big|_{\{x,y,z\}} = R_j\big|_{\{x,y,z\}}$ for all $i, j \in K$, then $R_0\big|_{\{x,y,z\}} \in \mathscr{R}^*\big|_{\{x,y,z\}}$.

To prove this claim, let $z \in X \setminus \{x, y\}$ and let $\mathbf{R} \in \mathscr{R}^N$ satisfying (a), (b), and (c). From the independence of irrelevant alternatives axiom we can assume, without loss of generality, that the set $\{x, y, z\}$ is top-ranked in each R_i and that $R \in \mathscr{R}^{*N}$. Let $\mathbf{R}^\circ \in \mathscr{R}^N$ such that (i) $R_i^\circ = R_i$ for all $i \in K$, (ii) $yP_i^\circ x$, $xP_i^\circ z$, and $zP_i^\circ y$ for all $i \notin K$, and (iii) $\mathbf{R} \not\succcurlyeq \mathbf{R}^\circ$. Because xD_Ky it follows that $xP_0^\circ y$.

From condition (c) it follows that there are two cases: either $xP_i^\circ z$ for all $i \in K$, or $zR_i^\circ x$ for all $i \in K$. In the former case, $xP_i^\circ z$ for all $i \in N$, which implies (by weak Pareto), that $xP_0^\circ z$. Because $xP_0^\circ y$ and $xP_0^\circ z$, it follows that $R_0^\circ\big|_{\{x,y,z\}} \in \mathscr{R}^*\big|_{\{x,y,z\}}$. In the latter case, $zP_i^\circ y$ for all $i \in N$, which implies (by weak Pareto), that $zP_0^\circ y$. Because $xP_0^\circ y$ and $zP_0^\circ y$, it follows that $R_0^\circ\big|_{\{x,y,z\}} \in \mathscr{R}^*\big|_{\{x,y,z\}}$. Because $R_0^\circ\big|_{\{x,y,z\}} \in \mathscr{R}^*\big|_{\{x,y,z\}}$ it follows from monotonicity and independence of irrelevant alternatives that $R_0\big|_{\{x,y,z\}} \in \mathscr{R}^*\big|_{\{x,y,z\}}$, proving the claim.

Step two. Let $\mathbf{R}' \in \mathscr{R}^{*N}$ such that, for all $i \in K$, $xP_i'y$ and $yP_i'z$ and, for all $i \notin K$, $yP_i'x$ and $yP_i'z$. Because xD_ky it follows that $xP_0'y$, and because $yP_i'z$ for all $i \in N$ it follows from weak Pareto that $yP_0'z$. Because $\mathbf{R}'$ satisfies requirements (a), (b), and (c) of step one, it follows that $R_0'\big|_{\{x,y,z\}} \in \mathscr{R}^*\big|_{\{x,y,z\}}$ and therefore $xP_0'z$. By the independence of irrelevant alternatives axiom, this implies that $x\bar{D}_Kz$. In other words:

$$xD_Ky \text{ implies that } x\bar{D}_Kz. \tag{1}$$

Now, let $\mathbf{R}'' \in \mathscr{R}^{*N}$ such that, for all $i \in K$, $zP''_i x$ and $xP''_i y$ and, for all $i \notin K$, $zP''_i x$ and $yP''_i x$. Because $xD_k y$ it follows that $xP''_0 y$, and because $zP''_i x$ for all $i \in N$ it follows from weak Pareto that $zP''_0 x$. Because $\mathbf{R}''$ satisfies requirements (a), (b), and (c) of step one, it follows that $R''_0 \big|_{\{x,y,z\}} \in \mathscr{R}^* \big|_{\{x,y,z\}}$ and therefore $zP''_0 y$. By the independence of irrelevant alternatives axiom, this implies that $z\bar{D}_K y$. In other words:

$$xD_K y \text{ implies that } z\bar{D}_K y. \tag{2}$$

By interchanging y and z in statement (2) it follows that $xD_K z$ implies that $y\bar{D}_K z$, and by replacing x by y, y by z, and z by x in statement (1) it follows $yD_K z$ implies that $y\bar{D}_K x$. As a consequence, it follows that

$$xD_K y \text{ implies that } y\bar{D}_K x. \tag{3}$$

By interchanging x and y in statements (1), (2), and (3), it follows that $yD_K x$ implies that $y\bar{D}_K z$, $z\bar{D}_K x$, and $x\bar{D}_K y$. As a consequence, we are led to the implication that for every $\{x, y, z\} \subseteq X$, if $xD_K y$ then $a\bar{D}_K b$ for every $a, b \in \{x, y, z\}$. If $|X| = 3$ this completes the proof.

If $|X| \geq 4$, then let $w \in X \setminus \{x, y, z\}$. By replacing y with z and z with w in statement (2), it follows that $xD_K z$ implies that $w\bar{D}_K z$, which concludes the proof.[11]

□

Proof of Theorem 1. Let f be a collective choice rule that satisfies the monotonicity, weak Pareto, independence of irrelevant alternatives, and non-dictatorship axioms. We will derive a contradiction.

Let $S \subseteq N$ be a decisive coalition of minimal size, so that $|T| < |S|$ implies that $xD_T y$ is false for all $x, y \in X$. By the weak Pareto axiom, such a coalition S exists. By the non-dictatorship axiom and Lemma 1, $|S| \geq 2$. Without loss of generality, let $xD_S y$. Let $S_1 \subseteq S$ such that $|S_1| = 1$, let $S_2 \equiv S \setminus S_1$, and let $S_3 \equiv N \setminus S$.

Let $\mathbf{R} \in \mathscr{R}^{*N}$ be a transitive profile such that (a) $xP_i y$, $yP_i z$, and $xP_i z$ for all $i \in S_1$, (b) $zP_i x$, $xP_i y$, and $zP_i y$ for all $i \in S_2$, and (c) $yP_i z$, $zP_i x$, and $yP_i x$ for all $i \in S_3$.

Let $R_\times \in \mathscr{R}$ such that $xP_\times y$, $yP_\times z$, and $zP_\times x$. Let $R_+ \in \mathscr{R}$ such that $xP_+ z$, $zP_+ y$, and $yP_+ x$.

Let $\mathbf{R}^A, \mathbf{R}^B, \mathbf{R}^C \in \mathscr{R}^N$ be profiles such that (a) $R^A_i = R^B_i = R^C_i = R_\times$ for all $i \in S_1$, (b) $R^A_i = R^B_i = R^C_i = R_+$ for all $i \in S_2$, and (c) $R^A_i = R_\times$, $R^B_i = R_+$, and $R^C_i = R_i$ for all $i \in S_3$.

Because of the independence of irrelevant alternatives axiom, we can assume, without loss of generality, that the elements $x, y, z \in X$ are top-ranked in profiles $\mathbf{R}$, $\mathbf{R}^A$, $\mathbf{R}^B$, and $\mathbf{R}^C$ and that $\mathbf{R} \big|_{X\setminus\{x,y,z\}} = \mathbf{R}^A \big|_{X\setminus\{x,y,z\}} = \mathbf{R}^B \big|_{X\setminus\{x,y,z\}} = \mathbf{R}^C \big|_{X\setminus\{x,y,z\}}$. It follows that $\mathbf{R} \not\succcurlyeq \mathbf{R}^A$, $\mathbf{R} \not\succcurlyeq \mathbf{R}^B$, and $\mathbf{R} \not\succcurlyeq \mathbf{R}^C$. Suppose, contrariwise, that R_0 is not transitive. It follows from monotonicity that neither R^A_0, R^B_0, nor R^C_0 may be

[11] We thank an anonymous referee for simplifying the proof.

transitive. Because S_2 is not a decisive coalition, it follows that $xR_0^A y$, $yR_0^A z$, and $zR_0^A x$. Because R_0^A is not transitive it follows that $S_1 \cup S_3$ must be decisive for at least one of the three pairs x over y, y over z, or z over x. By Lemma 1, it follows that $xD_{S_1 \cup S_3} y$ for all $x, y \in X$.

Because S_1 is not a decisive coalition, it follows that $xR_0^B z$, $zR_0^B y$, and $yR_0^B x$. Because R_0^B is not transitive it follows that $S_2 \cup S_3$ must be decisive for at least one of the three pairs x over z, z over y, or y over x. By Lemma 1, it follows that $xD_{S_2 \cup S_3} y$ for all $x, y \in X$.

Because $xD_{S_1 \cup S_3} y$ for all $x, y \in X$ it follows that $yP_0^C z$ and $zP_0^C x$. Because $xD_{S_2 \cup S_3} y$ for all $x, y \in X$ it follows that $yP_0^C x$. Therefore, R_0^C is transitive, which is a contradiction that proves that R_0 must be transitive.

By assumption, the coalition $S = S_1 \cup S_2$ is decisive for x over y. This implies that $xP_0 y$. Because $zP_i y$ only for $i \in S_2$ and S_2 is not decisive, it follows that $yR_0 z$. Because R_0 is transitive, it follows that $xP_0 z$. But this means that $xD_{S_1} z$, which implies, by Lemma 1, that S_1 is a dictator. This violates the non-dictatorship axiom, and concludes the impossibility proof.

Independence of the Axioms. We describe four collective choice rules. Each of the rules satisfies three of the axioms while violating the fourth. This is sufficient to prove the independence of the axioms.

Rule 1. For all $x, y \in X$, let $xR_0 y$ if and only if $|\{i \in N : xR_i y\}| \geq |\{i \in N : yR_i x\}|$. This rule clearly satisfies weak Pareto, independence of irrelevant alternatives, and non dictatorship, but violates monotonicity.

Rule 2. Let $d \in N$. For all $x, y \in X$, let $xR_0 y$ if and only if $xR_d y$. This rule clearly satisfies monotonicity, weak Pareto, and independence of irrelevant alternatives, but violates non-dictatorship.

Rule 3. Let $\mathscr{R}^T$ be the set of preference relations such that $R \in \mathscr{R}^T$ and $R' \succcurlyeq R$ implies that $R' \in \mathscr{R}^T$. If $R_1, R_2 \in \mathscr{R}^T$, let $f(R_1, ..., R_n) = R_1$, otherwise, let $f(R_1, ..., R_n) = R_2$. This rule satisfies monotonicity, weak Pareto, and non dictatorship, but violates independence of irrelevant alternatives.

Rule 4. For all $x, y \in X$, let $xR_0 y$. This rule clearly satisfies monotonicity, independence of irrelevant alternatives, and nondictatorship, but violates weak Pareto. □

3 Conclusion

This paper departs from the standard approach to preference aggregation in three ways. First, in light of research indicating that individual behavior may violate standard assumptions of rationality, we modify the standard model of preference aggregation to study the case in which neither individual nor collective preferences are required to satisfy transitivity or other coherence conditions. Second, we introduce the concept of an ordinal rationality measure which can be used to compare preference relations in terms of their level of coherence. Third, using this measure, we introduce a monotonicity axiom which requires that the collective preference become more rational when the individual preferences become more rational. We show that for any ordinal rationality measure, it is impossible to find a collective choice rule which satisfies the monotonicity axiom and the other standard assumptions introduced by Arrow (1963): unrestricted domain, weak Pareto, independence of irrelevant alternatives, and nondictatorship.

A natural question involves the extent to which the monotonicity axiom introduced in this paper substitutes for the standard assumption of transitivity. For example, consider an axiom, "transitive-to-transitive" which requires that every profile of transitive preference relations must map to a transitive social relation.[12] There is no logical relation between this axiom and the monotonicity axiom we propose. For example, a constant rule that maps all profiles to the same non-transitive social preference satisfies monotonicity but not this axiom. To see that a rule may satisfy the transitive-to-transitive axiom but not monotonicity, consider a rule in which the social preference coincides with that of the first agent when that agent, and only that agent, has a non-transitive preference relation, and which otherwise maps to a fixed transitive social preference. When all agents' preferences are transitive, this rule will lead to a transitive social preference, and thus it satisfies the transitive-to-transitive axiom. If the first agent's preferences change and become non-transitive, the social preference will clearly become non-transitive. However, if a second agent's preferences change and become non-transitive, the social preference will change back to the original transitive preference, thus violating monotonicity.

However, in the presence of weak Pareto, independence of irrelevant alternatives, and non-dictatorship, the two axioms are equivalent. To see this, note that in the context of Arrow (1963), the transitive-to-transitive axiom implies that Arrow's condition of unrestricted domain is satisfied on the set of transitive profiles. Consequently, when combined with weak Pareto and independence of irrelevant alternatives, this axiom implies the existence of an individual d who is decisive over every pair of alternatives for every transitive profile. That is xP_dy implies xP_0y for every transitive profile. By the independence of irrelevant alternatives axiom, however, it becomes irrelevant whether the profile is transitive; and hence individual d is a dictator.

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