Repeated bargaining

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Abstract

Two symmetric players bargain over an infinite stream of pies. There is one exogenously given pie in every period, whose size is stochastic, and the pies are iid. Play can be in a *tabula rasa* mode or *dispute* mode. When it is in the former, Nature selects a proposer and a responder with equal probabilities, and a proposal is made by the proposer regarding the division of the present pie. If there is agreement then it is implemented and play moves on to the next period, where it is again in tabula rasa. If there is rejection then dispute starts, which means that the players start to bargain over the disputed pie according to Rubinstein's (1982) protocol. As long as the disputed pie is Rubinstein-bargained over, all the new pies that arrive are not available for consumption (they disappear right after they materialize). Once a dispute settles, the game shifts back to tabula rasa. I characterize the game's unique stationary subgame perfect equilibrium.

Key Words: Bargaining; Repeated games.

JEL Codes: D70; D74.

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1 Introduction

Bargaining often takes place repeatedly, among the same players. Examples abound: firms and labor unions bargain periodically over wages, working conditions, health benefits, and more; firms bargain from time to time on how to cooperate, or collude; government officials bargain repeatedly over matters of policy, political influence, and so on.

In this paper I study a stylized model which is designed to focus on two aspects of repeated non-cooperative bargaining. First, the issues over which the players bargain arrive sequentially and stochastically. Second, resolution of earlier issues is necessary for reaping the benefits of future bargains. That is, as long as the bargaining problem of period t has not been solved, the players are not available to bargain over the surplus that materializes at period $t+1$. Furthermore, I make the assumption that if the problem of period t has not been solved by the beginning of period $t+1$, then the surplus of period $t+1$ disappears. Namely, bargaining opportunities have to be exploited as soon as they become available, otherwise they are lost. To illustrate the idea, think of two partners in a firm who need to decide whether to enter a certain market, in which operation is profitable only if one is a monopolist. If the partners can't decide fast, because there are some unsettled issues over which they are negotiating, some other competitor will fill in the vacuum, and the opportunity of profit-generation will be lost.

In the model, two players bargain repeatedly over divisions of pies. The pies are exogenous, and one pie "arrives" in the beginning of every period; the pies are stochastic and iid. A proposer and responder are selected by Nature, with equal probabilities, as long as no disagreements (i.e., rejections of offers) have occurred.[1] Once an offer is rejected, the game enters a "dispute phase" where the players bargain *a la* Rubinstein (1982) over the disputed pie. During the Rubinstein bargaining over

[1]This symmetric probabilistic method for selecting a proposer in bargaining games has first been studied by Binmore (1987).

this pie, all the new pies that arrive cannot be utilized, and they disappear instantly. Once an agreement over a disputed pie is reached, play reverts back to the original phase, in which Nature randomly selects a proposer and a responder in the beginning of every period.

The game has a unique stationary subgame perfect equilibrium. In equilibrium, there is agreement in every period (i.e., disputes never occur). Moreover, there is a cutoff such that if the pie size is below it, then the proposer demands (and obtains) the entire pie. When the pie size is above the cutoff it is split non-trivially between the players according to some formula. The reason for the "dictatorial demands of small pies" is that the responder does not find rejection of such dictatorial offers worthwhile, because by rejecting a proposal he triggers the risk of losing a large social surplus in the following round.

Bargaining games with a stochastic pie-size have been studied in the literature (see, e.g., Merlo and Wilson 1995), but no previous paper has combined—to the best of my knowledge—stochastic pies and a repeated-bargaining setting. One feature of my model that does have a counterpart (though a somewhat remote counterpart) in other works is that disagreement in period t affects the set of feasible utility-pairs of period $t + 1$. For instance, Fershtman and Seidmann (1993) studied an alternating offers game (over the division of a single pie) under the behavioral assumption that once a player rejects a share x of the pie, he commits to refusing to anything less than x in all future periods. The intertemporal links in my model are very different. Finally, my model can be viewed as complementary to the *bargaining and markets* literature, which considers models with a stochastic population of bargainers, but deterministic social surplus (see the book by Osborne and Rubinstein 1990; see in particular chapter 6 and the references therein).

The rest of the paper is organized as follows. Section 2 describes the model formally. The equilibrium is derived and discussed in Section 3. Section 4 concludes.

2 Model

Time is discrete and runs as $t = 1, 2, \cdots$. Two players bargain over an infinite stream of pies, one in every period. In each period t a pie of size m_t materializes, where m_t is a random draw from the distribution F over $[0, M]$.[2] Pies of different periods are independent of one another. Let $m^e \equiv \int_0^M t dF(t)$.

In any given period play can be in one of two phases: *tabula rasa* (TR) or *dispute* (D). In the first period, $t = 1$, the game starts at TR. In the beginning of each period t in which the game is at TR, Nature selects, randomly, a *proposer* and a *responder*. Each player has probability $\frac{1}{2}$ to be assigned to either role. Once the roles are assigned, that period's pie, m_t, materializes. The proposer offers a split of the pie: he asks for himself a share s and offers the remaining $m_t - s$ to the responder, where $s \in [0, m_t]$. If the responder accepts then this division of m_t is implemented, and the game moves to period $t+1$, where it is again at TR and the above interaction repeats. If, on the other hand, the responder rejects the offered split, then the game moves to period $t+1$, but it now starts in phase D. Now, the player who rejected the proposed division makes a counter offer. The pie of period $t+1$, m_{t+1}, disappears. That is, disagreement in period t entails a one-period delay and a loss of the next period's social surplus. The players bargain over the disputed pie, m_t, according to Rubinstein's (1982) alternating-offers protocol: every rejection entails a one-period delay and a change of roles between the proposer and the responder. As long as this Rubinstein-bargaining over the disputed pie takes place, all the new pies that arrive disappear instantly. If there is perpetual disagreement over the disputed pie then the game ends with its outcome being a concatenation of the history that preceded bargaining over the disputed pie, and the perpetual disagreement that came after the disputed pie arrived. If there is agreement after τ rounds of Rubinstein-bargaining, then the agreed-upon split of m_t is implemented in period $t + \tau$, and play moves to period $t + \tau + 1$, where it starts at phase TR.

[2] Abusing notation a little, I use m_t to denote both the pie of period t as well as its size.

To make the above description formal, and in order to define strategies, I now define *histories.* The following notation will be useful. Given a number $m > 0$, let $D(m)$ be the set of all possible divisions of m between the players, namely $D(m) \equiv \{x \in \mathbb{R}_+^2 : x_1 + x_2 = m\}$. The initial history is $h_0 = \emptyset$. A history of length $t \geq 1$ is defined given a sequence of realizations of pie sizes, $\mathbf{m}_t = (m_1, \cdots, m_t)$. The set of those histories is denoted $H(\mathbf{m}_t)$. An element $h_t \in H(\mathbf{m}_t)$ takes the form $h_t = (h_t^1, \cdots, h_t^t)$, where for each $k \in \{1, \cdots, t\}$ the component $h_t^k = (h_t^k(1), h_t^k(2), h_t^k(3))$ belongs to the product set $D(z_k) \times \{1, 2\} \times \{Accept, Reject\}$, where the definition of $\{z_k\}_{k=1}^t$ is given by 1-3, and where condition $\star$ is satisfied:

1. $z_1 = m_1$,

2. For all $k < t$: $h_t^k(3) = Accept \Rightarrow z_{k+1} = m_{k+1}$, and

3. For all $k < t$: $h_t^k(3) = Reject \Rightarrow z_{k+1} = z_k$.

Condition $\star$: for all $k < t$, if $h_t^k(3) = Reject$ then $h_t^{k+1}(2) = 3 - h_t^k(2)$.

A history h_t records for each period up to t what was the pie to bargain over in this period and what was the offered split (namely, $h_t^k(1)$), who was the proposer (namely, $h_t^k(2)$), and what was the responder's response (namely, $h_t^k(3)$). The set of all histories of length t is $H_t \equiv \cup_{\mathbf{m}_t \in [0,M]^t} H(\mathbf{m}_t)$. The set of all histories is $H \equiv \cup_t H_t$.

A *strategy for player* i is a pair of functions, $\sigma^i = (d^i, r^i)$. Given a period $t+1$ ($t \geq 0$) and a history that leads to it, h_t, the *demand function* d^i specifies an offer as a function of the history and of the current pie, $d^i(h_t, m_{t+1}) \in [0, z_{t+1}]$. The *response function* r^i prescribes an accept/reject decision for any offer the opponent may make, as a function of the history and the offer made. A player uses the d^i component of his strategy whenever he is called to be the proposer, and uses the r^i component when he is called to be the responder.

Given a history h_t and a player i, let $\pi^i(h_t) = (\pi_1^i(h_t), \cdots, \pi_t^i(h_t))$ be the sequence of pie slices that i obtains along this history. These slices are defined in the obvious way.

For all $k \in \{1, \cdots, t\}$:

- $[h_t^k(2) = i]\&[h_t^k(3) = Accept] \Rightarrow \pi_k^i(h_t) = h_t^k(1)_i$.[3]
- $[h_t^k(2) \neq i]\&[h_t^k(3) = Accept] \Rightarrow \pi_k^i(h_t) = h_t^k(1)_{3-i}$.
- $[h_t^k(3) = Reject] \Rightarrow \pi_k^i(h_t) = 0$.

A strategy-pair $\sigma = (\sigma^1, \sigma^2)$ induces a probability distribution on H. Given the history h_t and its continuation $h_{t+\tau}$, the probability of arriving $h_{t+\tau}$ given h_t and given that σ is played is denoted by $Pr_\sigma(h_{t+\tau}|h_t)$. The set of continuation histories of h_t is denoted by $C(h_t)$. Player i's expected utility from continuation play given h_t and σ is:

$$U_i(\sigma|h_t) = \sum_{h_{t+\tau} \in C(h_t)} Pr_\sigma(h_{t+\tau}|h_t)\delta^\tau \pi^i_{t+\tau}(h_{t+\tau}),$$

where $\delta \in (0, 1)$ is a discount factor.

The strategy-pair σ is a *subgame perfect equilibrium* (equilibrium, for short) if σ^1 maximizes $U_1((., \sigma^2)|h_t)$ for all h_t, and the analogous requirement holds for player 2. An equilibrium σ is *stationary* if for each player i the strategy σ^i is independent of histories. That is, each player utilizes the same demand and response functions, no matter what is the history that prevailed before he was called to make his move. In what follows, I consider only stationary equilibria. I use $x(.)$ to denote player 1's demand function in such an equilibrium, and use $y(.)$ to denote player 2's demand function. Note that in a stationary equilibrium a player's expected utility is the same, no matter the history after which it is calculated. Denote i's expected utility by V_i. This number is called i's *equilibrium value* (value, for short).

Denote the above game by G.

[3] $h_t^k(1)_i$ is the i-th coordinate of $h_t^k(1)$.

3 The result

Theorem 1. *The game G has a unique stationary equilibrium. The equilibrium demand function, which is common to both players, is given by:*

$$d^*(m) = \begin{cases} m & \text{if } m \leq \frac{m^e}{2}, \\ \frac{m+\delta\frac{m^e}{2}}{1+\delta} & \text{otherwise.} \end{cases}$$

Every offer is accepted immediately on the path. In general (i.e., on and off the path), when the pie size is m the offer $m - x$ is accepted if and only if $m - x + \delta\frac{m^e}{2(1-\delta)} \geq \delta d^*(m) + \delta^2\frac{m^e}{2(1-\delta)}$.

Theorem 1 will be proved by several lemmas. In their proofs, Pi is a shorthand for "player i."

Lemma 1. *In a stationary equilibrium every offer is accepted immediately.*

Proof. Consider, wlog, period t, in which P1 is the proposer and the pie size is m. Let $(x, m-x)$ be the proposal. Assume by contradiction that this offer is rejected by P2. There are two possibilities: either P2's counter offer is accepted in the next round, or it is rejected. The latter cannot happen in a stationary equilibrium, because it leads with certainty to $t+2$, in which P1 is the proposer and the pie size is m. By stationarity, there is rejection in all future periods, which is impossible in a stationary equilibrium. Therefore P2's counter-offer, say $(m-y, y)$, is accepted at $t+1$. But then P1 has a profitable deviation at t: to offer $(m-y, y)$, which would be accepted because of subgame perfection. □

Lemma 2. *In a stationary equilibrium, whenever the responder is offered a strictly positive share of the pie, he is indifferent between accepting and rejecting it.*

Proof. By Lemma 1 the responder accepts the offer, hence he weakly prefers acceptance to rejection. Had this preference been strict, the proposer could have ϵ-increased his demand, and for a sufficiently small $\epsilon > 0$ the responder would still agree. □

Lemma 3. *In a stationary equilibrium, the players' values are equal; namely, $V_1 = V_2$.*

Proof. Consider a stationary equilibrium, and assume by contradiction, wlog, that $V_2 < V_1$. Since, by Lemma 1, there is acceptance in every round, and since each player has a 50% chance of becoming the proposer whenever a new pie arrives (and the game is at TR), it follows that there is a pie size $m > 0$ given which P2 is more generous than P1. That is, $y(m) < x(m)$. In particular, $y(m) < m$, so P2 offers a strictly positive share to P1. Hence, by Lemma 2, P1 is indifferent between acceptance and rejection:

$$m - y(m) + \delta V_1 = \delta x(m) + \delta^2 V_1.$$

Also, due to immediate acceptance, P2 weakly prefers acceptance of P1's offer to rejection of this offer:

$$m - x(m) + \delta V_2 \geq \delta y(m) + \delta^2 V_2.$$

Thus, we obtain:

$$y(m) + \delta x(m) = m + \delta(1-\delta)V_1 > m + \delta(1-\delta)V_2 \geq \delta y(m) + x(m),$$

implying that $y(m) > x(m)$—a contradiction.

□

I therefore denote the value of a stationary equilibrium by V.

Lemma 4. *Consider a stationary equilibrium whose value is V. If the pie size m satisfies $m \leq (1-\delta)V$ then the proposer demands (and obtains) the entire pie. If $m > (1-\delta)V$, then the proposer proposes a strictly positive share of the pie to the responder.*

Proof. Wlog, consider period t and suppose that P1 is the proposer. Consider first the case $m \leq (1-\delta)V$. Assume by contradiction that P1 demands in equilibrium

$x < m$. By Lemma 1, this offer is accepted by P2. Consider the deviation where instead of demanding for himself x, P1 demands the entire pie. P2 must reject, else the demand x is inconsistent with equilibrium. Now, there are two possibilities:

(1) There is agreement in $t+1$ regarding P2's counter offer. Note that the maximal period-$(t+1)$ payoff that P2 can obtain is m. Therefore, his rejection in t implies that $\delta m + \delta^2 V > \delta V$, or $m > (1-\delta)V$, which is impossible.

(2) There is disagreement in $t+1$. Then play moves to $t+2$ in which, again, P1 needs to propose a division of m. By the stationarity of the equilibrium he offers $(x, m-x)$, which is accepted. Therefore, the rejection of P2 at t implies $\delta^2(m-x) + \delta^3 V > \delta V$, or $\delta(m-x) + \delta^2 V > V$. Therefore $\delta m > (1-\delta^2)V$, and since $m \leq (1-\delta)V$ we obtain $\delta(1-\delta)V > (1-\delta^2)V$, or $\delta > 1+\delta$.

Consider now $m > (1-\delta)V$. Assume by contradiction that P1 demands the entire pie. By Lemma 1, P2 agrees to this demand in equilibrium. Consider the subgame that is played given the rejection of P1's offer; namely, the subgame that starts at $t+1$ in which P2 proposes a division of m. P1 would accept any x that satisfies $x + \delta V \geq \delta m + \delta^2 V$ (the RHS is his payoff in case he rejects P2's offer, due to the equilibrium's stationarity). In particular, he would agree to $x = \delta m - \delta(1-\delta)V$ (note that the RHS is positive per the assumption on m). I argue that the above deviation of P2 is profitable; namely, that

$$\delta\{m - [\delta m - \delta(1-\delta)V] + \delta V\} > \delta V.$$

The above simplifies to $m > (1-\delta)V$, which holds by assumption.

□

Lemma 5. *Consider a stationary equilibrium whose value is V. If $m > (1-\delta)V$ then the proposer's demand is independent of the proposer's identity.*

Proof. Assume by contradiction that there is an $m > (1-\delta)V$ such that when P1 offers a split of m he asks $x(m)$ for himself and when P2 offers such a split he asks $y(m) \neq x(m)$. By Lemma 4 $x(m), y(m) < m$ and therefore, by Lemma 2, the responder is indifferent between the rejection and acceptance of $m - z(m)$, for each $z(m) \in \{x(m), y(m)\}$. Therefore

$$m - x(m) + \delta V = \delta y(m) + \delta^2 V,$$

and

$$m - y(m) + \delta V = \delta x(m) + \delta^2 V.$$

Therefore $m + \delta(1-\delta)V = x(m) + \delta y(m) = y(m) + \delta x(m)$. Since $\delta \in (0,1)$, $x(m) = y(m)$—a contradiction. □

Lemma 6. *Consider a stationary equilibrium whose value is V. The proposer's demand function is given by:*

$$d_V(m) = \begin{cases} m & \text{if } m \leq (1-\delta)V, \\ \frac{m+\delta(1-\delta)V}{1+\delta} & \text{otherwise.} \end{cases}$$

Proof. We already saw in Lemma 4 that if $m \leq (1-\delta)V$ then the proposer demands the entire pie for himself. Consider then $m > (1-\delta)V$. By Lemma 4, the proposer offers a strictly positive pie share to the responder, and therefore, by Lemma 2, the responder is indifferent between its acceptance and rejection. Denoting the equilibrium offer by $(x, m-x)$, we see that Lemma 5 implies that $m - x + \delta V = \delta x + \delta^2 V$, hence $x = \frac{m+\delta(1-\delta)V}{1+\delta}$, as argued. □

Lemma 7. *The value of a stationary equilibrium is $V = \frac{m^e}{2(1-\delta)}$.*

Proof. By Lemmas 1 and 6, the value satisfies the following equation:

$$V = \frac{1}{2}\{\int_0^{(1-\delta)V} tdF(t)t + \int_{(1-\delta)V}^{M} d_V(t)dF(t)\} + \\ + \frac{1}{2}\int_{(1-\delta)V}^{M} [t - d_V(t)]f(t)dF(t) + \delta V,$$

which is uniquely solved by $V = \frac{m^e}{2(1-\delta)}$. □

Theorem 1 follows from Lemmas 1, 6, and 7.

The following properties of the equilibrium are verified straightforwardly. First, the demand converges to the entire pie as $\delta \to 0$. Second, the proposer always demands more than half the pie. Third, the demand converges to the average of the pie-size and the equilibrium's *per-period-equivalent* value (namely $\frac{m^e}{2}$) as $\delta \to 1$.

4 Conclusion

Within an economy to which a stochastic pie arrives once a period, I have considered a game that has two distinct phases: "business as usual" and "dispute." As long as business is as usual a proposer and responder are selected every period randomly and bargaining over the current pie takes place; once disagreement occurs play moves to the dispute-phase, where the disputed pie is Rubinstein-bargained over, and all the new pies that arrive during the dispute are lost. The unique stationary equilibrium is such that when the pie size is below a certain cutoff the proposer demands the entire pie, and otherwise the pie is split non-trivially between the players. This result is intuitive: when there is only a small amount at stake, the player who happens to have the "temporary monopoly power" can exploit it, and the other player won't object since objection is too costly in terms of the losses of future social surplus that it can potentially cause. When a lot is at stake, the responder would not agree to such greedy behavior since the potential for current payoff is too large to ignore, and since the cost of dispute is small relatively to the present surplus that needs to be divided.

I have focused on stationary equilibria. I suspect that the game does not have a non-stationary subgame perfect equilibrium, but I do not have a proof for this conjecture. Another open problem is the generalization of the result to the case where the players' discount factors differ. Several of the arguments involved in the Lemmas relay on the players' symmetry, and generalizing them is not straightforward. Such a generalization remains, at present, a task for future research.

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References

[1] Binmore, K., 1987. Perfect equilibria in bargaining models. The Economics of Bargaining, Basil Blackwell, Oxford.

[2] Fershtman, C. and Seidmann, D.J., 1993. Deadline effects and inefficient delay in bargaining with endogenous commitment. Journal of Economic Theory, **60**, pp.306-321.

[3] Merlo, A. and Wilson, C., 1995. A stochastic model of sequential bargaining with complete information. *Econometrica*, **63**, pp.371-399.

[4] Osborne, M.J. and Rubinstein, A., 1990. Bargaining and markets. Academic press.

[5] Rubinstein, A., 1982. Perfect equilibrium in a bargaining model. *Econometrica*, **50**, pp.97-109.