REGULARITY OF A GENERAL EQUILIBRIUM IN A MODEL WITH INFINITE PAST AND FUTURE

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Abstract. We develop easy-to-verify conditions assuring that comparative statics in a general equilibrium model where time is a real line is feasible, i.e., the implicit function theorem is applicable. Consider an equilibrium equation, $\Upsilon(k, E) = k$ of a model where an equilibrium variable (k) is a continuous bounded function of time, real line, and the policy parameter (E) is a locally integrable function of time. The key conditions are time invariance of the equilibrium map Υ and the requirement that the Fourier transform of the derivative of the map Υ with respect to the equilibrium variable k does not return unity. Further, in a general constant-returns-to-scale production and homogeneous life-time-utility overlapping generations model we show that the first condition is satisfied at a balanced growth equilibrium and the second condition is satisfied for "almost all" policies that give rise to such equilibria.
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1. Introduction

Policy evaluation in a general equilibrium model with overlapping generations is perceived as a daunting task, and our objective here is to provide a transparent way to make it more reliable. It is common, even in the case of a partial equilibrium analysis, to first look for equilibrium using numerical methods, then calibrate or estimate the parameters of the model to fit a given data set, and only then to perform policy "experiments", i.e., to evaluate the reaction of equilibrium variables given the calibrated parameters to a "shock" in policy, see, for example, [1] for the description of the algorithm. In this paper we work with policies that are not limited to one-time shocks, rather, we allow them to be functions of time and of individual characteristics. We offer the ground work for an analytical approach: as in the classical textbook case à la Debreu, we show that implicit function theorem can be applied to an equilibrium equation ($\Upsilon(k, E) = k$) for "almost any" distribution of endowments (E) that yields a balanced growth equilibrium, in which case the equilibrium variables (e.g., k) can be viewed as smooth functions of endowments (E), thought of as a transfer policy.[1] Thus, the equilibrium responses generated by such comparative statics are data-independent and can be directly calculated from the specification of the model.

The analytical approach we offer can be viewed as complimentary to the numerical one. Establishing local uniqueness (regularity of equilibria) validates the numerical approach by assuring that the close-by equilibrium after the policy change

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[1] See [11] for the overview of the existing applications of IFT to equilibria in infinite economies and the reasons why they can not be applied to the overlapping generations models.

is unique; moreover, analytical approximation for the reaction of equilibrium to the policy change provides a robustness check for the parallel numerical calculation.

We find that the key sufficient condition for regularity of equilibria is *time-invariance*, implying, in particular, that time in the model should be the whole real line. Its alternative, the half-line, or assuming existence of some "starting point" at zero, prevents one from properly modelling perfect foresight, cf. [2], and might be responsible for indeterminacy, cf. [7], [4], and hence inability to predict the effects of a policy. With time-invariance, verifying conditions needed for the implicit function theorem amounts to checking differentiability of the equilibrium map (Υ) and calculating Fourier transform of the derivative with respect to the endogenous variable (k).

The paper consists of two main parts. First we formulate the result for an arbitrary fixed point equation and then illustrate our approach using an overlapping generations model with a general smooth constant-returns-to-scale technology and assuming only time-separability and homogeneity of life-time consumption.[2] This rather minimal set of assumptions allows for balanced growth equilibria, as is well-known in the literature [8]. Such an equilibrium is a base-line equilibrium, to which the policy change is applied. We demonstrate how to verify our sufficient conditions in this case and, in addition, show that comparative statics is feasible for "almost any" transfer policy, i.e., the set of policies that is open and everywhere dense in the corresponding Banach space of policies.

Our argument rests on some building blocks from [11], where regularity was demonstrated for a parametric version of such an overlapping generations economy with Cobb-Douglas production.

2. The first formulation of the problem

Let P,V be the space of parameters and variables correspondingly, both are Banach spaces. In the example that follows the space of parameters (transfers) is the space of locally integrable functions and the equilibrium variable (capital path) is a continuous uniformly bounded function defined on the real line (time).

Assume that for a given model, equilibrium conditions can be reduced to a fixed-point equation in variable $k \in V$, given parameters $E \in P$,

$$F(k, E) \stackrel{\text{def}}{=} \Upsilon(k, E) - k = 0 \tag{1}$$

The objective of the modeller is to assure that one can evaluate the equilibrium response of k to the change in the policy parameter E at some equilibrium point (k^0, E^0).

To illustrate the idea, let us view E as a function of two real variables: an individual characteristic (age) and time. The initial, base-line equilibrium will be stationary in which E^0 is constant with respect to time, though it can still be a function of individual age, thus describing some fixed system of transfers across individuals of different ages, for example as in a pay-as-you-go pension system. Then the change in the policy parameter, δE, describes how the transfer system is altered across ages and over time. The resulting reaction of equilibrium variables, in particular capital path, is what we want to calculate.

For finite economies, such calculations are done by appealing to an implicit function theorem (IFT), which requires differentiability of F and invertibility of the derivative $\frac{\partial F}{\partial k}$ at an equilibrium point (k^0, E^0). We will follow the same route here. First, let us formulate the suitable version of the implicit function theorem. The differentiability below is in the sense of Fréchet.

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[2]Such models are indispensable if one is to evaluate the effect of a policy that involves an intergenerational transfer, such as a pension reform for example, cf. e.g., [3] for the overview.

Theorem 1 ([14], Thm. 3.8.5.)**.** *Let $\mathscr{E},\mathscr{F},\mathscr{G}$ be three Banach spaces, g be a continuously differentiable map from an open set $O \subset \mathscr{E} \times \mathscr{F}$ into $\mathscr{G}$, $F\colon (k,E) \mapsto F(k,E)$. Let (a,b) be a point in O, $F(a,b)=c$.*

If $\frac{\partial F}{\partial k}(a,b)$ is invertible in the space of linear operators from $\mathscr{F}$ to $\mathscr{G}$, then there exist opens sets $A \subset \mathscr{E}$ and $B \subset \mathscr{F}$, $A \times B \in O$ such that for every $E \in B$, there is a unique solution (in k) of the equation $F(k,E)=c$ which belongs to A and there is a continuously differentiable function $k=\phi(E)$ from B to A such that $F(\phi(E),E)=c$. Its derivative is given by

$$\phi'(b) = -\Big(\frac{\partial F}{\partial k}(a,b)\Big)^{-1} \circ \Big(\frac{\partial F}{\partial E}(a,b)\Big) \tag{2}$$

Remark 1. Notice that in the notation of Theorem 1, $\frac{\partial F}{\partial k}(k,E)$ is invertible in a neighbourhood of (a,b), since $(k,E) \mapsto \frac{\partial F}{\partial k}(k,E)$ is a continuous map invertible at $(k,E)=(a,b)$. It follows that there exists a neighbourhood N of b such that $\frac{\partial F}{\partial k}(\phi(E),E)$ is invertible for any $E \in N$.

Now our task is two-fold.

The first task is to find sufficient conditions for $\frac{\partial F}{\partial k}(k^0,E^0)$ to be invertible.

The second task is to show that invertibility holds for a *large* set of parameters, i.e., for an open dense subset of P. That the set is open follows from remark 1 given differentiability of F, cf. section 4.6.

3. The first task

Our first task is to assure invertibility of the derivative of F, which defines the equilibrium condition (1), with respect to the endogenous variable (k) at the baseline equilibrium, k^0.

The condition we offer is a combination of the two results that will be presented next. The main requirement that they impose is *time-* or *translation-invariance*.

The first result is a particular case of a theorem in [6]. It establishes that any bounded translation-invariant operator from $L^\infty(\mathbb{R})$ to $L^\infty(\mathbb{R})$ is given by a convolution with a finite measure.[3]

Let us start with some notation.

Definition 1. (i) Let $t_h\colon t \mapsto t+h$ be the translation by h on $\mathbb{R}$; and $\mathbb{S}_h\colon \xi \mapsto \xi \circ t_{-h}$ be the time-shift on functions of time.

(ii) For $f \geq 0$ Lebesgue-measurable and a bounded measure μ the *convolution* $\mu \star f$ is the equivalence class of $t \mapsto \int \tilde{f}(t-s)\mu(ds)$ for any $\tilde{f} \in f$.

Here we will formulate the result for a closed subspace of L^∞, denoted by C_b, which is the space of continuous uniformly bounded functions. In the example that follows any equilibrium k belongs to that space.

Theorem 2 (Hörmander)**.** *If A is a bounded translation-invariant operator from C_b to C_b then there is a unique finite measure μ such that*

$$A\varphi = \mu \star \varphi \quad \forall \varphi \in C_b. \tag{3}$$

To see how this result can be applied and is relevant to our case, consider again an equilibrium k^0 that solves $\Upsilon(k,E^0)=k$ for some exogenous policy E^0. The derivative of Υ with respect to k at the equilibrium point (k^0,E^0) is a linear operator from the space of bounded continuous functions, C_b, to itself, mapping a perturbation δk to the resulting change in the value of Υ. It is this derivative that we want to be translation-invariant in order to apply the Hörmander's theorem. As

[3] $L^\infty(\mathbb{R})$ is the set of uniformly bounded real-valued functions on $\mathbb{R}$. The reference to $\mathbb{R}$ is omitted when no ambiguity arises by doing so.

we will see later, it will be easy to prove time-invariance of the derivative at a point where the policy variable, E, is itself time-invariant. But for now let us formulate the immediate consequence of the theorem.

Corollary 1. *If Υ is differentiable and $\Big(\$_h \frac{\partial \Upsilon}{\partial k}(k^0, E^0)\Big)(\delta k) = \Big(\frac{\partial \Upsilon}{\partial k}(k^0, E^0)\Big)(\$_h \delta k)$ for any $\delta k \in C_b$ and any $h > 0$, then there is a finite measure k^K such that for any $\delta k \in C_b$*

$$\Big(\frac{\partial \Upsilon}{\partial k}(k^0, E^0)\Big)(\delta k) = \mathrm{k}^K \star \delta k \tag{4}$$

Proof. Boundedness of the derivative operator follows by its definition and then the result follows from theorem 2. ∎

So, in case the hypothesis of the corollary is satisfied, i.e., the equilibrium map is translation-invariant, then the derivative $\frac{\partial \Upsilon}{\partial k}(k^0, E^0)$ is given by a convolution kernel k^K. Coming back to the original task, it is still not clear how this result will help us find a way to verify the invertibility of

$$\frac{\partial F}{\partial k}(k^0, E^0) = \frac{\partial \Upsilon}{\partial k}(k^0, E^0) - I, \text{ where } I \text{ is the identity operator.} \tag{5}$$

Definition 2.
(i) For a bounded measure μ, its *Fourier transform* (FT) is $\widehat{\mu}(\omega) = \int e^{\mathrm{i}\omega t}\mu(dt)$ ($\widehat{g}$ for $g \in L_1$).
(ii) A spectrum of an operator T from a Banach space to itself is the set $\{z \in \mathbb{C} \mid T - zI \text{ is not invertible}\}$.

In the view of the definition of the spectrum, our task now is to assure that unity is not in the spectrum of $\frac{\partial \Upsilon}{\partial k}(k^0, E^0)$, viewed as an operator. Fortunately, there is an easy way to do so. If the convolution kernel k^K is absolutely continuos with respect to Lebesgue measure on $\mathbb{R}$ (i.e., is given by a function: the measure k^K is of the form $g(x)d\mu(x)$, where g is a function and μ is a Lebesgue measure), then we can use Wiener's theorem (lemma), reproduced below. The theorem implies that the spectrum of a convolution operator which is given by such a k^K is a closure of the set of the values returned by its Fourier transform. In this case the invertibility of $\frac{\partial F}{\partial k}(k^0, E^0)$ is equivalent to the requirement that the Fourier transform of k^K does not return unity.

Here is the Wiener's theorem. For a function $f \in L^1(\mathbb{R})$ define a bounded operator $B_f \colon L^\infty \to L^\infty$ by

$$B_f \colon g \mapsto f * g$$

Theorem 3 (Wiener)**.** *The spectrum of B_f is $\{\widehat{f}(\omega) \mid \omega \in \mathbb{R}\} \cup \{0\} \subset \mathbb{C}$.*

Thus, the first task is completed. The summary of the argument is in figure 1.

4. The second task

The second task is to show that the conditions that we formulated in the previous section are not too demanding, i.e., they hold for a large subset of parameters. Such a statement can be vacuous in the absence of any model, so let us set one up first. As we do so, we will also illustrate how to use the approach outlined in section 3.

First, we describe the model, define the equilibrium map Υ and show that it satisfies the assumptions: differentiability at a base-line equilibrium with time invariant derivative that is given by a function, cf. the three red boxes in figure 1. We will then proceed to our main task here, presenting the main result of the paper, theorem 4.

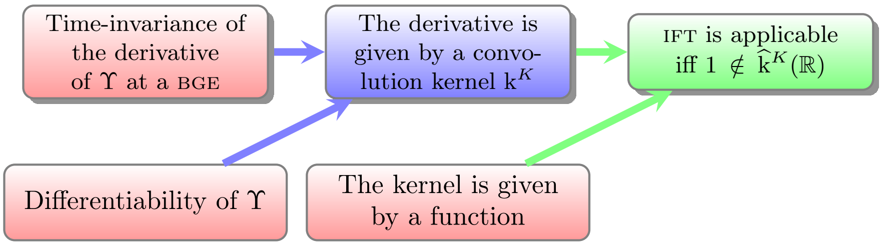

FIGURE 1. A schematic representation of the argument. The purple arrows correspond to our corollary to Hörmander's theorem, the green arrows are based on Weiner's theorem and the IFT. The assumptions are colored red.

4.1. **The overlapping generations model.** Consider a generalisation of OG model from [10], with constant returns to scale production and time-separable, positively homogeneous life-time utility.

Life span of any individual born at $x \in \mathbb{R}$ is $[0, 1]$. His life-time utility is

$$U_x = \int_0^1 e^{-\beta s} u(\hat{c}_{x,s}) ds$$

which is defined over the set of individual consumption plans $\hat{c}_{x,s}$, $\overline{\mathbb{R}}_+$-valued Lebesgue-measurable functions of age s for every x, given felicity $u\colon \mathbb{R}_+ \to \mathbb{R}$.

Before we continue, let us note that without loss of generality we can assume now that the felicity, u, is of a familiar form, $u(z) = \frac{z^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}$, for $\sigma \neq 1$. Here is why.

Proposition 1. *If the time-separable life-time utility is homogeneous of a fixed degree, then the instantaneous felicity is a power function, i.e., if for any life-time consumption plan* $\hat{c}\colon [0, 1] \to \mathbb{R}_+$, *there is a* $\rho \in \mathbb{R}$ *such that for any* $n > 0$, $U(n\hat{c}) = n^\rho U(\hat{c})$, *and if* U *can be written as*

$$U(\hat{c}) = \int_0^1 e^{-\beta s} u(\hat{c}_s) ds \tag{6}$$

for some real-valued function u, *and* $\beta \in [0, 1]$, *then* $u(x) = Ax^\rho$ *for some* $A > 0$.

Proof. By the assumptions, for any $n > 0$ there is a real number ρ such that for any life-time consumption plan $\hat{c}$,

$$\int_0^1 e^{-\beta s} u(n\hat{c}_s) ds = n^\rho \int_0^1 e^{-\beta s} u(\hat{c}_s) ds \tag{7}$$

Pick a constant life-time plan $\hat{c}_s = 1$, $s \in [0, 1]$. Then (7) implies $u(n) = n^\rho u(1)$, let now $A = u(1)$, which completes the proof. ■

Individuals have two sources of income: work and transfers. Profits are zero since the production satisfies constant-returns to scale. Life-cycle labor efficiency is an arbitrary non-negative function of age ζ_s, $s \in [0, 1]$, and it is zero beyond the life-span. There is no disutility from labor.

The second source of income for an individual born at x is a consumption transfer $\omega_{x,s}$ given at any age $s \in [0, 1]$. ω is *locally integrable*, but is not necessarily positive. Transfers are the exogenous (policy) parameter of the model.

Credit markets are unrestricted, and individuals have no bequest motive. So the only relevant constraint is the life-time budget constraint. Denote by M_x the total income of an individual born at x,

$$M_x \overset{\text{def}}{=} \int_0^1 (p^C_{x+s}\omega_{x,s} + w_{x+s}\zeta_s)ds$$

where the price of consumption goods is p^C, and price for a unit of efficient labour is w_t, both $\overline{\mathbb{R}}_+$-valued Lebesgue-measurable functions of time.

Population grows exponentially at rate ν: $N_x dx \overset{\text{def}}{=} N_0 e^{\nu x}dx$ individuals are born in $[x, x+dx]$ for any $x \in \mathbb{R}$, $N_0 > 0$.

Aggregate total *productive labour* available at t is:

$$L_t = N_0 e^{\gamma t}\int_{t-1}^t \zeta_{t-x}e^{\nu x}dx = N_0 e^{(\gamma+\nu)t}\int_0^1 \zeta_s e^{-\nu s}ds \tag{8}$$

where γ is the per-capita productivity growth.

It will be convenient to work with normalised variables: $E_{t,s}$ is the total endowment (resp. consumption) per unit of productive labour at time t of individuals of age s, Ω_t is the aggregate normalised endowment at t, and φ_s is the measure of individual labour efficiency.

Notation 4.1. (i) $E_{t,s} = \frac{N_{t-s}\omega_{t-s,s}}{L_t}$, $\Omega_t = \int_0^1 E_{t,s}ds$; $\varphi_s = \frac{e^{-\nu s}\zeta_s}{\int_0^1 e^{-\nu u}\zeta_u du}$.

We impose a very mild assumption on the transfers by *not* requiring them to be even summable over all times, hence including a perpetual transfer in or out of the economy. The only requirement we impose is, roughly, a local integrability one: the volume of transfers to agents of all ages ($\int |E_{\cdot,s}|ds$) summed over a unit interval of time, $t \in [x, x-1]$, is uniformly bounded (over $x \in \mathbb{R}$).

To formulate the assumption, first define a norm $\|\|_{\infty,1}$ for real-valued functions on $\mathbb{R}$.

Definition 3. For $h\colon \mathbb{R} \to \mathbb{R}$, let $\|h\|_{\infty,1} = \mathsf{sup}_x \int_{x-1}^x |h(t)|dt$. Let $L_{\infty,1} = \{h\colon \mathbb{R} \to \mathbb{R}\colon \|h\|_{\infty,1} < \infty\}$.

Assumption 1. $\|\int |E_{\cdot,s}|ds\|_{\infty,1} < \infty$.[4]

All firms (investment and manufacturing) are competitive and have finite lives.

Manufacturing firms produce consumption and investment goods using a constant-returns-to-scale instantaneous production function $G(K_t, L_t)$, $F\colon \mathbb{R}^2_+ \to \mathbb{R}_+$ concave, continuous, positively homogeneous of degree 1, and differentiable on $\mathbb{R}^2_{++}$. $f(x) \overset{\text{def}}{=} G(x, 1)$. The production set is any closed convex cone with free-disposal, containing the graph of the production function and the activities of transforming output into consumption and investment, and contained in the closed convex cone spanned by the production function, free-disposal, and 2-way transformations of output into consumption and investment. Note that this formulation includes some possible restrictions on reversibility of production processes, for example, expressing whether consumption can be turned into an investment good.

By homogeneity of F, f is strictly concave, and in addition, we impose the following mild assumptions, cf. [10, assumption 4].

Assumption 2. $\mathsf{lim}_{x\to\infty} \frac{f'(x)}{x} < R$, and $\exists x \in \mathbb{R}\colon f(x) > Rx$.

The technology of an investment firm is the capital accumulation equation

$$K'_t = I_t - \varrho K_t, \tag{9}$$

where I_t is the investment and $\varrho > 0$ is the depreciation factor, along with the initial condition:

[4] In what follows we will refer to this assumption (somewhat loosely) by saying that E belongs to $L_{\infty,1}$.

Assumption 3 (Initial Condition)**.** For any feasible path K, $e^{(\varrho - f'_\infty)t}K_t$ converges exponentially to 0 at $-\infty$, where $f'_\infty \stackrel{\text{def}}{=} \lim_{x\to\infty} \frac{f(x)}{x}$.

The word "exponentially" can be dropped here iff $\int_1^\infty \frac{f(x)-xf'_\infty}{x^2}dx < \infty$.

Alternatively, one could represent the differential equation 9 in its integral form. It is shown in [10, cor. 2] that feasibility restrictions imply that the aggregate capital accumulation equation can be written using the Lebesgue integral,

$$K_t = e^{-\varrho t}\int_{-\infty}^{t} I_s e^{\varrho s} ds \tag{10}$$

The investment firms have finite life-spans and rent capital at a market rate of r_t per unit to the manufacturing firms. The assumptions imposed on the investment sector are exactly the same as in [10, sect. 2.2.3].

Further, we normalise all the aggregates and introduce shortcuts.

Notation 4.2. (i) $k_t = \frac{K_t}{L_t}$, $y_t = \frac{Y_t}{L_t}$, $i_t = \frac{I_t}{L_t}$, $c_t = \frac{C_t}{L_t}$.
(ii) $R = \gamma + \nu + \varrho$, $\mathfrak{r}_t = R - f'(k_t)$.
(iii) $\eta = (\gamma + \nu)(1 - \sigma) + \beta\sigma$

Define a *reduced* economy as in [10], in which all the aggregates are the normalised aggregates ($L_t = 1$, $\gamma = \nu = 0$), labour efficiency is φ and transfers are E. The new β is η/σ and the new ϱ is R. By [10, lemma 1] a *reduced* economy is isomorphic to the original one and henceforth we will analyse equilibria of the reduced economy.

4.2. **Equilibria of the model.** To stress, we consider a *competitive equilibrium* in its classical form with all the agents choosing the best allocations given the prices and the ownership rights, while all the markets clear. In particular, the *material balance* condition is satisfied:

$$c_t + i_t \le y_t + \Omega_t \quad \forall t \in \mathbb{R} \tag{11}$$

We choose a balanced growth equilibrium, defined below, as a base-line, at which the policy will be evaluated.

Definition 4. (i) Endowments are *stationary* if

$$E_{t,s} = E^0_s = \frac{e^{-(\gamma+\nu)s}\omega_s}{\int_0^1 \zeta_s e^{-\nu s} ds}$$

and so $\Omega = \int E^0_s ds$.

(ii) A *balanced growth equilibrium* (BGE) is an equilibrium of an economy with stationary endowments, such that K_t is an exponential function of time.

(iii) A BGE is a *golden rule equilibrium* (GRE) if $\forall t, f(k_t) - Rk_t = \max_z(f(z) - Rz)$.

In [11] it was shown that in a BGE, K_t can only grow at a rate $\gamma + \nu$, so the normalised capital path, k, is constant in time in such an equilibrium.

Our equilibrium characterisation, however, has to include non-stationary equilibria as well because after the initial policy E^0 is changed by δE a new equilibrium might emerge which is not necessarily a BGE. In the view of the proposition 1 above, the model is now identical to the one analysed in [10], from whence we borrow the characterisation of equilibriua.

It is shown in [10, prop. 1] that the normalised capital path k_t has to be uniformly bounded from above, which follows from the assumptions imposed on production function, capital accumulation condition and material balance. The capital accumulation equation, $k_t = e^{-Rt}\int_{-\infty}^{t} i_s e^{Rs} ds$, also implies that k is a continuous function of time. Hence $k \in C_b$, thus justifying the choice of space for equilibrium variable in section 3.

In this paper we will focus on an open subset $C_{b,0}$ of C_b where the initial capital path resides (at a BGE, as will be shown below). This subset is defined as follows.

$$C_{b,0} \stackrel{\text{def}}{=} \{k \in C_b \mid \exists \epsilon > 0 \text{ such that } k_t \geq \epsilon \ \forall t \in \mathbb{R}\}$$

Next, let us define the map whose fixed point, by proposition 2, is, indeed a candidate equilibrium. For that we need to add another assumption, which we maintain for the remainder of the paper.

Assumption 4. f is twice continuously differentiable on $(0, \infty)$.

Definition 5. Define a map $\Upsilon\colon C_{b,0} \times L_{\infty,1} \to C_b$, $(k, E) \mapsto \tilde{k}$, as a composition:

(i) $k \mapsto y\colon y_t = f(k_t)$
(ii) $k \mapsto \mathfrak{r}\colon \mathfrak{r}_t = R - f'(k_t)$
(iii) $(y, \mathfrak{r}, E) \mapsto c$
$c_t = \int_0^1 e^{-\eta u - \sigma \int_{t-u}^t \mathfrak{r}_s ds} \mathscr{B}_{t-u} du$
$\mathscr{B}_x = \frac{\mathscr{N}_x}{\mathscr{D}_x}$
$\mathscr{N}_x = \int_0^1 e^{\int_x^{x+s} \mathfrak{r}_t dt} [E_{x+s,s} + \varphi_s(f(k_{x+s}) - k_{x+s} f'(k_{x+s}))] ds$
$\mathscr{D}_x = \int_0^1 e^{-\eta s + (1-\sigma) \int_x^{x+s} \mathfrak{r}_t dt} ds$
(iv) $(y, E, c) \mapsto i\colon i_t = y_t + \int_0^1 E_{t,s} ds - c_t$
(v) $i \mapsto \tilde{k}\colon \tilde{k}_t = e^{-Rt} \int_{-\infty}^t e^{Rs} i_s ds$

Lemma 1. *In definition 5* $c, \tilde{k}, y, \mathfrak{r} \in C_b$, $i \in L_{\infty,1}$, *so* Υ, *indeed, maps to* C_b.

Proof. The two conditions, f being continuously differentiable and $k \in C_{b,0}$, imply $y \in C_{b,0}$. f being twice continuously differentiable by assumption 4 and $k \in C_{b,0}$ imply $\mathfrak{r} \in C_b$. $\mathscr{D}$ is bounded away from zero by $\int_0^1 e^{-s(|\eta| + |1-\sigma| \operatorname{sup}_t |\mathfrak{r}|)} ds > 0$ and from above by $\int_0^1 e^{s(|\eta| + |1-\sigma| \operatorname{sup}_t |\mathfrak{r}|)} ds$. Similarly, $\mathscr{N}$ is uniformly bounded from above and continuity of $\mathscr{N}$ and $\mathscr{D}$ follows from their definitions. Then also $\mathscr{B}, c$ are in C_b by equation (iii) in the definition of Υ. Conditions (i), (iv) applied with $c = 0$ and (v) imply $\operatorname{sup}_t |\tilde{k}_t|$ is uniformly bounded from above and yield $i_t \in L_{\infty,1}$. Hence by equation (v), $\tilde{k}$ is continuous, and thus $\tilde{k} \in C_b$. ■

Now let us use the equilibrium characterisation from [10] combining corollaries 11 and 12 there. In the formulation below "general model" refers to the model presented above and "basic model" is the one with an additional assumption requiring full reversibility (between consumption, output, and investment), as is standard in a single-physical-commodity case.

Proposition 2 ([10])**.** *Let* $k \in C_{b,0}$ *be a fixed point of the map* Υ, *i.e., a solution to* $\Upsilon(k, E) = k$ *for some* $E \in L_{\infty,1}$, *and such that* $\mathscr{N}_t \geq 0$ *for all* $t \in \mathbb{R}$. *Let the prices for investment, consumption and output be equal:* $p^I = p^C = p^Y = p$, *where*

$$p_t = p_0 e^{\int_0^t \mathfrak{r}_s ds} \tag{12}$$

and interest rate and wages be defined as follows:

$$r_t = p_t f'(k_t) \tag{13}$$

$$w_t = (f(k_t) - k_t f'(k_t)) p_t \tag{14}$$

Then the five conditions in definition 5 and the prices, interest rate and wages as defined above characterise all of the following:

(i) all equilibria in $C_{b,0}$ *of the general model, provided* $\Omega_t \leq 0 \ \forall t$ *(e.g., pure transfers),* $f'(0) = \infty$ *and* $f(0) = 0$ *and provided the production vector component of the solution belongs to the instantaneous production set for all t;*

(ii) *all equilibria in $C_{b,0}$ of the general model where $0 < i_t < y_t$ a.e., provided the solution satisfies $0 < i_t < y_t$;*
(iii) *if $f'(0) = \infty$ and k is constant, all* BGE *of the basic model;*
(iv) *if k is constant, all* BGE *of the general model with $\omega = 0$, $f(0) = 0$ and $f'(0) = \infty$.*

BGE admit a more elegant representation, to which we now turn. It will be used to evaluate the derivative of Υ at a base-line. The characterisation of BGE is from [10, corollary 13]. For simplicity we associate a BGE with its constant capital path.

Notation 4.3. $\Phi(x) \stackrel{\text{def}}{=} \frac{e^x-1}{x}$.

Corollary 2. *Assume $f'(0) = \infty$. The set of* BGE *of the basic model is the set of constant positive capital paths, $k^0 \in \mathbb{R}_+$, that satisfy*

$$k^0\mathfrak{r} = \int_0^1 m_s^0 ds - \frac{\Phi(-\varkappa)}{\Phi(\mathfrak{r}-\varkappa)} \int_0^1 e^{\mathfrak{r}s} m_s^0 ds \tag{15}$$

$$\text{with } \int_0^1 e^{\mathfrak{r}s} m_s^0 ds \geq 0,\ m_s^0 \stackrel{\text{def}}{=} E_s^0 + (f(k^0) - k^0 f'(k^0))\varphi_s,\ \varkappa \stackrel{\text{def}}{=} \eta + \sigma\mathfrak{r} \tag{16}$$

with the rest of the (constant) quantities determined by:
(i) $\mathfrak{r}^0 = R - f'(k^0)$
(ii) $y^0 = f(k^0)$
(iii) $i^0 = Rk^0$
(iv) $\mathscr{N}^0 = \int_0^1 e^{\mathfrak{r}^0 s} m_s^0 ds$
(v) $c^0 = \frac{\Phi(-\varkappa)}{\Phi(\mathfrak{r}^0-\varkappa)}\mathscr{N}^0$
(vi) $\mathscr{D}^0 = \Phi(\mathfrak{r}^0 - \varkappa)$, $\mathscr{B}^0 = \frac{\mathscr{N}^0}{\mathscr{D}^0}$

and the prices determined by the conditions (12)-(14) *of proposition 2.*

Note that since $k^0 > 0$, a constant capital path that returns k^0, trivially, belongs to $C_{b,0}$.

4.3. **Properties of the equilibrium map Υ.** Here we establish two key properties of the equilibrium map: time-invariance and differentiability. These properties enable us to verify the three assumptions needed to apply the method described in section 3, cf. figure 1. To do so, first we establish the time-invariance of the equilibrium map itself (which is easier) in proposition 3, and, as a consequence, assert in corollary 3 the time-invariance of the derivative at a BGE. Second, we prove the differentiability by calculating the (total) derivative of Υ explicitly in lemma 2 and use this calculation in lemma 3 to show that the derivative with respect to k at a BGE is given by a function as required.

4.3.1. *Time-invariance.* The time-invariance property of the equilibrium system expresses the idea that the equilibrium conditions do not change if the "calendar" changes: there is no condition that depends solely on time t: only the dynamics of the policy parameter and the interconnections between different equilibrium variables matter. To stress, such condition will in general be violated if time is only a half-line, in which case the date, t, is measuring the amount of time that passed since time zero, before which no adjustments (in anticipation of future policy changes) could be made.

Proposition 3. *Υ is time-invariant with respect to k, E, i.e.,*

$$\Upsilon(\$_h k, \$_h E) = \$_h \Upsilon(k, E) \tag{17}$$

where $\$_h k_t = k_{t-h}$ and $\$_h E_{t,s} = E_{t-h,s}$, $h \in \mathbb{R}$.

Proof. The first two maps in the definition of Υ are time-invariant by construction:

(i) $f(\$_h k) = \$_h y$
(ii) $R - f'(\$_h k) = \$_h \mathfrak{r}$

That the third map is shift-invariant,

(iii) $(\$_h y, \$_h \mathfrak{r}, \$_h E) \mapsto \$_h c$ follows from the identity

$$\int_x^{x+s} \$_h \mathfrak{r}_t dt = \int_{x-h}^{x-h+s} \mathfrak{r}_t dt = \$_h \int_x^{x+s} \mathfrak{r}_t dt \tag{18}$$

and by construction. Finally, the last two maps are time-invariant by construction:

(iv) $\$_h y + \int_0^1 \$_h E_{t,s} ds - \$_h c = \$_h i$
(v) $e^{-Rt} \int_{-\infty}^t e^{Rs} \$_h i_s ds = e^{-Rt} \int_{-\infty}^t -h e^{R(s+h)} i_s ds = \$_h \tilde{k}$

■

4.3.2. *Derivative of* Υ.

Lemma 2. *Υ is continuously Fréchet-differentiable from $C_{b,0} \times L_{\infty,1}$ to $C_{b,0}$.*

Proof. Here we use differentiability of the basic operations (product, sum, composition) as well as composition with a continuously differentiable function. These statements are proven in the on-line appendix [11, app. B2-B3] for a more general notion of differentiability. Hence what is left now is to prove differentiability of each separate map in the definition of Υ, to which we now turn.

(i) The derivative of the first map is $\delta k_t \mapsto \delta y_t = f'(k_t)\delta k_t$ which is well-defined by differentiability of f and is continuous as f'' exists.
(ii) Similarly, the derivative of the second map is $\delta k_t \mapsto \delta \mathfrak{r}_t = -f''(k_t)\delta k_t$, as f is twice differentiable and is continuous as f'' is continuous. Continuity of the derivatives of the following maps is by a direct calculation.
(iii) Now we want to show that there is a (multi-linear) map $(\delta k_t, \delta \mathfrak{r}_t, \delta E) \mapsto \delta c_t$, which is the derivative of the map (iii) in the definition of Υ, def. 5. Let us first analyse its basic elements. We start with the simplest, $\mathscr{D}$, which depends only on $\mathfrak{r}$ and not on the other two variables, E, y. Using the properties of the exponential, the derivative of $\mathscr{D}$ with respect to $\mathfrak{r}$ is a map

$$\delta \mathfrak{r}_x \mapsto \delta \mathscr{D}_x = (1-\sigma) \int_0^1 \{ \int_x^{x+s} \delta \mathfrak{r}_t dt \} e^{-\eta s + (1-\sigma) \int_x^{x+s} \mathfrak{r}_t dt} ds$$

Now define $m_{x,s} \stackrel{\text{def}}{=} E_{x+s,s} + \varphi_s (f(k_{x+s}) - k_{x+s} f'(k_{x+s}))$. Then $\mathscr{N}_x = \int_0^1 e^{\int_x^{x+s} \mathfrak{r}_t dt} m_{x,s} ds$. The derivative of m is a map

$$(\delta E_{x,s}, \delta k_x) \mapsto \delta m_{x,s} = \delta E_{x+s,s} - \varphi_s k_{x+s} f''(k_{x+s}) \delta k_{x+s}$$

So, the derivative of $\mathscr{N}$ is

$$(\delta \mathfrak{r}_x, \delta m_{x,s}) \mapsto \delta \mathscr{N}_x = \int_0^1 e^{\int_x^{x+s} \mathfrak{r}_t dt} \delta m_{x,s} ds + \int_0^1 \{ \int_x^{x+s} \delta \mathfrak{r}_t dt \} e^{\int_x^{x+s} \mathfrak{r}_t dt} m_{x,s} ds$$

Finally, $(\delta \mathscr{N}_x, \delta \mathscr{D}_x) \mapsto \delta \mathscr{B}_x = \frac{\delta \mathscr{N}_x}{\mathscr{D}_x} - \frac{\mathscr{N}_x \delta \mathscr{D}_x}{\mathscr{D}_x^2}$ and so the derivative of c is

$$(\delta \mathscr{B}_t, \delta \mathfrak{r}_t) \mapsto \delta c_t = \int_0^1 e^{-\eta u - \sigma \int_{t-u}^t \mathfrak{r}_s ds} \delta \mathscr{B}_{t-u} du - \sigma \int_0^1 \{ \int_{t-u}^t \delta \mathfrak{r}_s ds \} e^{-\eta u - \sigma \int_{t-u}^t \mathfrak{r}_s ds} \mathscr{B}_{t-u} du$$

(iv) $(\delta y, \delta c, \delta E_{t,s}) \mapsto \delta i_t = \delta y_t - \delta c_t + \int_0^1 \delta E_{t,s} ds$
(v) with $\xi_R(x) \stackrel{\text{def}}{=} \mathbb{1}_{x \geq 0} e^{-Rx} dx$

$$\delta \tilde{k} = \xi_R \star \delta i \tag{19}$$

■

4.4. Applying Hörmander's theorem to assure that $\frac{\partial \Upsilon}{\partial k}(k^0, E^0)$ is a convolution.

Corollary 3. *The derivative* $\frac{\partial \Upsilon}{\partial k}$ *at* (k^0, E^0), *where* k^0 *is a* BGE *for an economy with endowment* E^0, *is given by a convolution kernel* $\mathrm{k}^K_{E^0}$, *i.e., there is a finite measure* $\mathrm{k}^K_{E^0}$ *such that* $\delta\Upsilon(k^0, E^0) = \mathrm{k}^K_{E^0} \star \delta k$, *for any sufficiently small* $\delta k \in C_b$.

Proof. $\Upsilon(k, E^0)$ is time-invariant by proposition 3, using then the definition of a derivative, we get the time-invariance of $\frac{\partial \Upsilon}{\partial k}(k^0, E^0)$, where E^0 is constant in time. The conclusion then follows from corollary 1. ∎

4.5. Applying Weiner's theorem to assure applicability of IFT.

Next step is to explicitly compute the kernel $\mathrm{k}^K_{E^0}$, show it is absolutely continuous with respect to Lebesgue measure and calculate its Fourier transform in order to use Wiener's theorem, the last statement from section 3.

Lemma 3. *Pick a policy* E^0 *and a* BGE k^0 *corresponding to* E^0. *Let* $m^0_s = E^0_s + \varphi_s(f(k^0) - k^0 f'(k^0))$, $\Xi_v = \mathbb{1}_{0\le v\le 1} \int_v^1 e^{s\mathfrak{r}} m^0_s ds$.

Then kernel $\mathrm{k}^K_{E^0}$ *from corollary 3 is given by a function. Its Fourier transform satisfies*

$$\widehat{\mathrm{k}^K_{E^0}}(\omega) = \frac{f'(k^0) - \widehat{\mathrm{k}^C_{E^0}}(\omega)}{R - \mathrm{i}\omega} \qquad \omega \in \mathbb{R} \tag{20}$$

where, i *is the imaginary unit,* $\varkappa \stackrel{\text{def}}{=} \eta + \sigma\mathfrak{r}$ *and*

$$\mathrm{k}^C_{E^0} \stackrel{\text{def}}{=} \mathrm{k}^X \star \mathrm{k}^B_{E^0} + \mathrm{k}^Y_{E^0} \tag{21}$$

$$\mathrm{k}^X(s) \stackrel{\text{def}}{=} \mathbb{1}_{0\le s\le 1} e^{-s\varkappa} ds \tag{22}$$

$$\mathrm{k}^B_{E^0} \stackrel{\text{def}}{=} \frac{1}{\mathscr{D}^0}[\mathrm{k}^N_{E^0} - \frac{\mathscr{N}^0}{\mathscr{D}^0}\mathrm{k}^D] \tag{23}$$

$$\mathrm{k}^D(s) \stackrel{\text{def}}{=} \mathbb{1}_{0\le s\le 1} \frac{e^{\mathfrak{r}-\varkappa} - e^{(\mathfrak{r}-\varkappa)s}}{\mathfrak{r} - \varkappa} ds \tag{24}$$

$$\mathrm{k}^N_{E^0}(s) \stackrel{\text{def}}{=} -\mathbb{1}_{0\le s\le 1} f''(k^0)\Big[k^0 e^{s\mathfrak{r}}\varphi_s + \int_s^1 e^{s\mathfrak{r}} m^0_s ds\Big] ds \tag{25}$$

$$\mathrm{k}^Y_{E^0}(s) \stackrel{\text{def}}{=} \mathbb{1}_{0\le s\le 1} \frac{\mathscr{N}^0}{\mathscr{D}^0}\sigma f''(k^0) \frac{e^{-\varkappa} - e^{-s\varkappa}}{-\varkappa} ds \tag{26}$$

Proof. First, we evaluate $\frac{\partial \Upsilon}{\partial k}$, at (k^0, E^0), a BGE, following the same steps as in the proof of lemma 2.

(i) $\delta k_t \mapsto \delta y_t = f'(k^0)\delta k_t$
(ii) $\delta k_t \mapsto \delta \mathfrak{r}_t = -f''(k^0)\delta k_t$
(iii) $\big(\delta k_t, \delta\mathfrak{r}_t, E^0\big) \mapsto \delta c_t$

$$\begin{aligned}\delta\mathscr{D}_x &= -(1-\sigma)f''(k^0)\int_0^1 e^{(\mathfrak{r}^0-\varkappa)s}\{\int_x^{x+s} \delta k_t dt\} ds \\ &= \int \mathbb{1}_{0\le t-x\le 1} \frac{e^{\mathfrak{r}^0-\varkappa} - e^{(\mathfrak{r}^0-\varkappa)(t-x)}}{\mathfrak{r}^0 - \varkappa} \delta k_t dt = \mathrm{k}^D \star \delta k\end{aligned}$$

where k^D is as defined in the claim. Using the definition for m as in the proof of lemma 2, its derivative with respect to k is

$$(\delta k_x) \mapsto \delta m_{x,s} = -\mathbb{1}_{0\le s\le 1}\varphi_s k^0 f''(k^0)\delta k_{x+s}$$

Then,

$$\delta\mathscr{N}_x = -k^0 f''(k^0)\int_0^1 e^{s\mathfrak{r}}\varphi_s\delta k_{x+s}ds - f''(k^0)\int_0^1\{\int_x^{x+s}\delta k_t dt\}e^{s\mathfrak{r}}m_s^0 ds \tag{27}$$

$$= -f''(k^0)\int \mathbb{1}_{0\le z-x\le 1}\Big[k^0 e^{(z-x)\mathfrak{r}}\varphi_{z-x} + \Xi_{z-x}\Big]\delta k_z dz = \mathrm{k}^N_{E^0}\star\delta k \tag{28}$$

Further, $\delta\mathscr{B}_x = \frac{1}{\mathscr{D}^0}\big[\delta\mathscr{N}_x - \frac{\mathscr{N}^0\delta\mathscr{D}_x}{\mathscr{D}^0}\big] = \mathrm{k}^B_{E^0}\star\delta k.$
Thus, the derivative of c is

$$\delta c_t = \int_0^1 e^{-u\varkappa}\delta B_{t-u}du + \frac{\mathscr{N}^0}{\mathscr{D}^0}\sigma f''(k^0)\int_0^1\{\int_{t-u}^t\delta k_s ds\}e^{-u\varkappa}du \tag{29}$$

Note, $\int_0^1\{\int_{t-u}^t\delta k_s ds\}e^{-u\varkappa}du = \int_{t-1}^t \frac{e^{-\varkappa}-e^{-(t-s)\varkappa}}{-\varkappa}\delta k_s ds$. So,

$$\delta c_t = \mathrm{k}^X_{E^0}\star\delta B + \frac{\mathscr{N}^0}{\mathscr{D}^0}\sigma f''(k^0)\int \mathbb{1}_{0\le t-s\le 1}\frac{e^{-\varkappa}-e^{-(t-s)\varkappa}}{-\varkappa}\delta k_s ds \tag{30}$$

$$= \mathrm{k}^X_{E^0}\star\delta B + \mathrm{k}^Y_{E^0}\delta k = \mathrm{k}^X_{E^0}\star\mathrm{k}^B_{E^0}\star\delta k + \mathrm{k}^Y_{E^0}\star\delta k = \mathrm{k}^C_{E^0}\star\delta k \tag{31}$$

Finally, as in the proof of lemma 2, combining the last two derivatives, $(\delta y, \delta c)\mapsto \delta i_t = \delta y_t - \delta c_t$, and $\delta i \mapsto \delta\tilde{k} = \xi_R\star\delta i$, where $\xi_R(x) \stackrel{\text{def}}{=} \mathbb{1}_{x\ge 0}e^{-Rx}dx$, we get

$$\frac{\partial\Upsilon}{\partial k}(\delta k, E^0) = \xi_R\star(\delta y_t - \delta c_t) = \xi_R\star(f'(k^0)\delta k_t - \mathrm{k}^C_{E^0}\star\delta k) \tag{32}$$

$$= (f'(k^0)\xi_R - \xi_R\star\mathrm{k}^C_{E^0})\star\delta k = \mathrm{k}^K_{E^0}\star\delta k \tag{33}$$

where $\mathrm{k}^K_{E^0} \stackrel{\text{def}}{=} f'(k^0)\xi_R - \xi_R\star\mathrm{k}^C_{E^0}$, which demonstrates that this kernel is given by a function, as are all of its components.

Next, we calculate the Fourier transform of $\mathrm{k}^K_{E^0}$, which concludes the proof:

$$\widehat{\mathrm{k}^K_{E^0}} = f'(k^0)\widehat{\xi_R} - \widehat{\xi_R}\widehat{\mathrm{k}^C_{E^0}} = \widehat{\xi_R}(f'(k^0) - \widehat{\mathrm{k}^C_{E^0}}) = \frac{f'(k^0) - \widehat{\mathrm{k}^C_{E^0}}}{R - \mathrm{i}\omega} \tag{34}$$

■

At this point one can directly verify whether the range of $\mathrm{k}^K_{E^0}$ contains unity and hence check whether a given equilibrium is regular. For an example of such calculation see [11, sect. 8.2], applied to a particular parametrisation of the current model. The values of k that fail the test for $E^0 = 0$ are referred to as "critical points" on the graphs of BGE there.

This concludes the illustration of the approach developed in section 3.

4.6. **Completing task 2.** Now we can formulate the task more precisely: our goal now is to show that the "good set", i.e., the set of policies E^0 for which the range of $\widehat{\mathrm{k}^K_{E^0}}$ does not contain unity is open and dense in the set of all policies that generate a BGE, i.e., the space of integrable functions with support $[0,1]$, or equivalently, $L_1([0,1])$.

That the "good set" is *open* follows from lemma 2, definition of function F (equation (1)) and remark 1 following the IFT.

To build an argument proving that the "good set" is *dense*, let us pick an arbitrary initial policy E^0 that satisfies the non-negativity of consumption constraint, $\mathscr{N}^0 = \int_0^1 e^{\mathfrak{r}^0 s}m_s^0 ds > 0$ and find a solution $k^0 > 0$ to the fixed point equation, so that, using $\varkappa = \eta + \sigma\mathfrak{r}$

$$k^0\mathfrak{r} = \int_0^1 m_s^0 ds - \frac{\Phi(-\varkappa)}{\Phi(\mathfrak{r}-\varkappa)}\int_0^1 e^{\mathfrak{r}s}m_s^0 ds \tag{35}$$

where $\mathfrak{r} = R - f'(k^0)$ and $m_s^0 = E_s^0 + \varphi_s(f(k^0) - k^0 f'(k^0))$. Now, perturb E^0 by δE^0 so as to preserve the inequality $\int_0^1 e^{\mathfrak{r}^0 s}(m_s^0 + \delta E^0)ds > 0$ (decreasing $\|E^0\|$ if needed) and, more importantly, to keep k^0 constant, i.e., δE should satisfy

$$\int_0^1 \delta E_s ds = \frac{\Phi(-\varkappa)}{\Phi(\mathfrak{r} - \varkappa)} \int_0^1 e^{\mathfrak{r}s} \delta E_s ds \tag{36}$$

Let $H \subset L_1([0,1])$ be a set of perturbations δE defined by equation (36). This set includes "no perturbation", i.e., $\delta E = 0 \in H$. H is a hyperplane in $L_1([0,1])$ if $\mathfrak{r} \neq 0$, and is all of $L_1([0,1])$ if $\mathfrak{r} = 0$.[5]

Define the set of "bad points" as the set of policies for which the range of $\widehat{\mathrm{k}_{E^0}^K}$ contains unity, call it Z. The main result of this section, theorem 4, shows that around any point from that "bad set" Z, there is an arbitrarily close "good" policy point. To construct such a point, we pick one that gives rise to the same equilibrium capital path k^0, i.e., the change in E (that perturbs the kernel $\mathrm{k}_{E^0}^K$) belongs to H.

Since the "good set" is open the following theorem is sufficient to establish that the good set is *everywhere dense* in $L_1([0,1])$.

Theorem 4. *Let*

$$\begin{aligned} Z =& \{E \in L_1([0,1]) \colon \exists \omega \in \mathbb{R} \colon \widehat{\mathrm{k}_E^K}(\omega) = 1\} \\ H =& \{E \in L_1([0,1]) \colon \int_0^1 E_s ds = \frac{\Phi(-\varkappa)}{\Phi(\mathfrak{r} - \varkappa)} \int_0^1 e^{\mathfrak{r}s} E_s ds\} \end{aligned}$$

Then for any $E^0 \in Z$, and $\epsilon > 0$ there is $\delta E \in H$ such that $\|\delta E\| < \epsilon$ and $E^0 + \delta E \notin Z$.

The proof is in the appendix.

5. Conclusions and Further directions

In this paper we developed an easy-to-implement test for regularity of equilibria and validity of comparative statics for dynamic models where the parameter of the model can vary with time (e.g., a pension system). In other words, we offer a test to validate "policy experiments" around a given equilibrium: passing the test implies local uniqueness meaning that equilibrium variables can be represented as functions of the parameter in a neighbourhood of the base-line equilibrium. Our test applies to *translation-invariant* equilibrium (fixed-point) mappings, implying, in particular, that the model has to include the "infinite past." It is easy to apply the test to check regularity of stationary base-line equilibria. The test requires the derivative of the equilibrium map with respect to an equilibrium variable at the baseline to be given by a function and its Fourier transform not to return unity.

We illustrate our approach for a general OG model with transfer policies and show that "almost all policies" pass the test.

One could also be interested in applying comparative statics to dynamic games, with the caveat that the approach developed in this paper is for a single fixed point equation, so it could be applied to two-player games, for example, or to study aggregate (average) behaviour. The crucial condition is, again, time-invariance, which can be satisfied if the time horizon in the game is infinite in both directions, i.e., if there is no end and no beginning to the game. Such modelling strategy has its advantages, for instance, it allows representing truly forward-looking agents who can react to any future event as far in advance as necessary. Besides, the need to specify "period 0" conditions might eliminate some very "natural" candidates for equilibria that support cooperation in repeated games [9, ch.12.1.2]. Having verified that an IFT is applicable allows one to view the equilibria in the vicinity

[5] $\mathfrak{r} = 0$ in a GRE, cf. definition 4.

of a stationary base-line as functions of the parameters of the game. The same approach can also be used to access multiplicity of equilibria.

In addition, if one is ready to verify a stronger notion of differentiability (S^1) defined in [11, on-line appendix], our method can be used to establish stability of equilibria as well.

Appendix A. Proof of the main result

In order to prove theorem 4, we will state and prove several auxiliary statements.

Notation A.1. $\Psi(x,\omega) \overset{\text{def}}{=} \int_0^1 e^{\mathrm{i}\omega s} \frac{e^x - e^{xs}}{x} ds$.

Lemma 4. *Pick a policy E^0 and a* BGE *k^0 corresponding to E^0. Pick a $\delta E \in H$, let $\xi_s \overset{\text{def}}{=} \mathbb{1}_{0\le s\le 1} \int_s^1 e^{\mathfrak{r}v}(\delta E)_v dv$. Then, the perturbed Fourier transform $\widehat{\delta \mathrm{k}^C_{E+\delta E}}(\omega)$ as a result of a change δE in policy is*

$$\frac{\sigma\,\xi_0 f''(k^0)}{\Phi(\mathfrak{r}-\varkappa)}\Psi(-\varkappa,\omega) - \frac{\Phi(\mathrm{i}\omega-\varkappa)}{\Phi(\mathfrak{r}-\varkappa)}\Big[f''(k^0)\widehat{\xi}(\omega) + \Psi(\mathfrak{r}-\varkappa,\omega)\frac{\xi_0}{\Phi(\mathfrak{r}-\varkappa)}\Big] \tag{37}$$

Proof. Since $\delta E \in H$, the same k^0 that solves the equilibrium equation for E^0, solves it for $E^0 + \delta E$. By lemma 3, k^X does not depend on E^0. So, by the same lemma,

$$\delta \mathrm{k}^C_{E^0+\delta E} = \mathrm{k}^X \star \delta \mathrm{k}^B_{E^0+\delta E} + \delta \mathrm{k}^Y_{E^0+\delta E} \tag{38}$$

Further, by corollary 2, $\mathscr{D}^0 = \Phi(\mathfrak{r}-\varkappa)$ is independent of E^0. Then, by lemma 3,

$$\delta \mathrm{k}^Y_{E^0+\delta E}(x) = \frac{\sigma\,\xi_0 f''(k^0)}{\Phi(\mathfrak{r}-\varkappa)} \mathbb{1}_{0\le x\le 1} \frac{e^{-\varkappa} - e^{-x\varkappa}}{-\varkappa} \tag{39}$$

Again, by lemma 3, $\mathrm{k}^D(s)$ is independent of E^0. Therefore,

$$\delta \mathrm{k}^B_{E^0+\delta E}(s) = \frac{1}{\Phi(\mathfrak{r}-\varkappa)}\Big[\delta \mathrm{k}^N_{E^0+\delta E}(s) - \mathrm{k}^D(s)\frac{\xi_0}{\Phi(\mathfrak{r}-\varkappa)}\Big] \tag{40}$$

$$\delta \mathrm{k}^N_{E^0+\delta E}(s) = -f''(k^0)\mathbb{1}_{0\le s\le 1}\xi_s \tag{41}$$

Now we calculate the FT of $\delta \mathrm{k}^C_{E^0+\delta E}$. First, by equation (38),

$$\widehat{\delta \mathrm{k}^C_{E^0+\delta E}} = \widehat{\mathrm{k}^X}\,\widehat{\delta \mathrm{k}^B_{E^0+\delta E}} + \widehat{\delta \mathrm{k}^Y_{E^0+\delta E}} \tag{42}$$

Further, by definition of k^X in lemma 3,

$$\widehat{\mathrm{k}^X} = \int_0^1 e^{s(\mathrm{i}\omega-\varkappa)} ds = \Phi(\mathrm{i}\omega-\varkappa) \tag{43}$$

By equation (40) and using notation 4.3,

$$\widehat{\delta \mathrm{k}^B_{E^0+\delta E}} = \frac{1}{\Phi(\mathfrak{r}-\varkappa)}\Big[\widehat{\delta \mathrm{k}^N_{E^0+\delta E}} - \widehat{\mathrm{k}^D}\frac{\xi_0}{\Phi(\mathfrak{r}-\varkappa)}\Big] \tag{44}$$

Similarly, by equations (39) and (41), definition of k^D in lemma 3 and using notation A.1,

$$\widehat{\delta \mathrm{k}^Y_{E^0+\delta E}} = \xi_0 \frac{\sigma f''(k^0)}{\Phi(\mathfrak{r}-\varkappa)}\Psi(-\varkappa,\omega) \tag{45}$$

$$\widehat{\mathrm{k}^D} = \Psi(\mathfrak{r}-\varkappa,\omega) \tag{46}$$

$$\widehat{\delta \mathrm{k}^N_{E^0+\delta E}} = -f''(k^0)\widehat{\xi} \tag{47}$$

Substituting these values into equation (42) we get the result. ∎

The following lemma is from [12, Lemma 3.9.].

Lemma 5. *Let V be a real vector space and $l_1, \ldots, l_k$ be linear functionals on V. Let m be another linear functional on V such that $m(v) = 0$ whenever $l_1(v) = l_2(v) = \ldots = l_k(v) = 0$. Then m is a linear combination of $l_1, \ldots, l_k$.*

Lemma 6. *Let f be a real-valued function, such that $f(t)\,\mathsf{exp}(\lambda|t|) \in L^1(\mathbb{R})$ for $\lambda > 0$. Then the integral $\int_{-\infty}^{\infty} e^{\mathrm{i}\omega t} f(t)dt$ converges for $|\Im\omega| \le \lambda$ and the function $\widehat{f}(\omega) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} e^{\mathrm{i}\omega t} f(t)dt$ is complex-analytic in $\{\omega \in \mathbb{C} \mid |\Im\omega| < \lambda\}$.*

Proof. Convergence of the integral is immediate since $|e^{\mathrm{i}\omega t}| = e^{|\Im\omega||t|}$. To check the differentiability of $\widehat{f}$ at ω with $|\Im\omega| < \lambda$ note that

$$\lim_{h\to 0} \frac{e^{\mathrm{i}(\omega+h)t} - e^{\mathrm{i}\omega t}}{h} = \mathrm{i}te^{\mathrm{i}\omega t}$$

where $h \in \mathbb{C}$. Also, by Taylor's theorem we have

$$|e^{\mathrm{i}(\omega+h)t} - e^{\mathrm{i}\omega t} - \mathrm{i}the^{\mathrm{i}\omega t}| \le t^2|h|^2 e^{(|\Im\omega|+|\Im h|)|t|}$$

and hence

$$\left|\frac{e^{\mathrm{i}(\omega+h)t} - e^{\mathrm{i}\omega t}}{h} - \mathrm{i}te^{\mathrm{i}\omega t}\right| \le t^2|h|e^{(|\Im\omega|+|\Im h|)|t|}$$

Choose $0 < \epsilon < \lambda - |\Im\omega|$. For $|h| < \epsilon$, $t^2|h|e^{(|\Im\omega|+|\Im h|)|t|} \le Ce^{\lambda|t|}$ for some constant C, dependent on ϵ, and $|f(t)|t^2|h|e^{(|\Im\omega|+|\Im h|)|t|} \le C|f(t)|e^{\lambda|t|}$. Then the dominated convergence theorem implies that

$$\lim_{h\to 0} \frac{\widehat{f}(\omega+h) - \widehat{f}(\omega)}{h} = \lim_{h\to 0} \int_{-\infty}^{\infty} \frac{e^{\mathrm{i}(\omega+h)t} - e^{\mathrm{i}\omega t}}{h} f(t)dt = \mathrm{i}\int_{-\infty}^{\infty} te^{\mathrm{i}\omega t} f(t)dt,$$

and the statement follows. ∎

Proposition 4. *Assume that function h is complex-analytic in the strip $S = \{z \in \mathbb{C} \mid |\Im z| < C\}$, $C > 0$ and that $|h(z)| \to 0$ as $z \to \infty$ while $z \in \mathbb{R}$. Assume that equation $h(z) = 1$ has real solutions $t_1, t_2, \ldots, t_k$ such that t_j is of multiplicity m_j for $j \in \{1, \ldots, k\}$ and let $a_j = \lim_{z\to t_j} \frac{h(z)-1}{(z-t_j)^{m_j}}$. Let $g(z)$ be another complex-analytic function in S such that $|g(z)| \to 0$ as $z \to \infty$ while $z \in \mathbb{R}$ and satisfying*

$$\frac{g(t_j)}{a_j} \notin \mathbb{R} \quad \forall j \in \{1, \ldots, k\} \tag{48}$$

Then there exists $\epsilon_0 > 0$ such that for any real $\varepsilon > 0$ with $|\varepsilon| < \epsilon_0$ the equation $h_\varepsilon(z) = 1$, where $h_\varepsilon \stackrel{\text{def}}{=} h + \varepsilon g$, has no real solutions.

Proof. Notice that analyticity of h and the assumption $|h(z)| \to 0$ as $z \to \infty$ imply that the number of real roots of $h(\cdot) - 1$ is finite, as stated. In addition, since h is analytic, a_j, as defined in the statement, is the first (non-zero) coefficient of the Taylor series around the corresponding root, t_j, of $h(\cdot) - 1$, thus $a_j = \frac{h^{(m_j)}(t_j)}{m_j!}$, where $h^{(m_j)}$ is the m_j-th derivative of h.

Let $r > 0$ be such that for any $z \in \mathbb{R}$ with $|z| \ge r$, $|h(z)| < \frac{1}{2}$, $|g(z)| < \frac{1}{2}$. It follows that $h_\varepsilon - 1$ has no real roots z with $|z| \ge r$ for ε sufficiently small (say, $|\varepsilon| < 1$). Choose now a positive $c < C$ such that all the roots of $h - 1$ in the closure of $G = \{z \in \mathbb{C}\colon |\Im z| < c, |\Re z| < r\}$ are real (possible since by the analyticity of $h - 1$ it has finitely many roots in $\{z \in \mathbb{C}\colon |\Im z| < c, |\Re z| < r\}$); these are precisely t_j, $j = 1, \ldots, k$. For sufficiently small ε, $h_\varepsilon - 1$ has exactly as many roots in G as $h - 1$. This follows from Rouché's theorem.[6] By construction, $h - 1$ has no roots on the boundary ∂G of G. It follows that $|h - 1|$ is bounded away from zero on the

[6] Cf. e.g., [13, 10.43 b]: If f and g are holomorphic on some bounded region G, with continuous boundary ∂G, and if with $|g(z)| < |f(z)|$ on ∂G, then f and $f + g$ have the same number of zeros in G.

boundary, since ∂G is a compact set. Further, $|g(z)|$ is bounded on ∂G. Therefore, if ε is small enough ($|\varepsilon| < \frac{\inf_{\partial G}|h(z)-1|}{\sup_{\partial G}|g(z)|}$) then the assumption of Rouché's theorem is satisfied, $\inf_{z\in\partial G}|h(z)-1| > \varepsilon \sup_{z\in\partial G}|g(z)|$.

An analogous argument using Rouché's theorem shows the following. Fix $\delta > 0$, $\delta < 1/2 \min_{j\neq j'}|t_j - t_{j'}|$. Then there is a real $\kappa > 0$ (depending on δ) such that for any $|\varepsilon| < \kappa$ the function $h_\varepsilon(z) - 1$ has precisely m_j roots in the disk $|z - t_j| < \delta$ for any j.

So, pick a root, $t_j + x$ of $h_\varepsilon(\cdot) - 1$ with $|x| < \delta$. We have $h(t_j + z) = 1 + z^{m_j}\phi(z)$, where $\phi(z)$ is an analytic function, $\phi(0) = a_j \neq 0$. Choose now $\delta > 0$, which is sufficiently small so that $m = \inf_{|z|<\delta} |\phi(z)| > 0$. Set also $M = \max_{|z|<\delta} |g(t_j + z)|$. Since $1 = h(t_j + x) + \varepsilon g(t_j + x)$, we have

$$x^{m_j}\phi(x) = -\varepsilon g(t_j + x)$$

and hence

$$|x|^{m_j} \leq K\varepsilon \text{ where } K = \frac{M}{m}.$$

Using the Taylor expansion,

$$|h(t_j + x) - (1 + a_j x^{m_j})| \leq C_1|x|^{m_j+1}$$
$$|g(t_j + x) - g(t_j)| \leq C_2|x|$$

for some C_1, C_2 independent of ε.

Returning to the equation $1 = h(t_j + x) + \varepsilon g(t_j + x)$, and using the above inequalities, we have

$$|a_j x^{m_j} + \varepsilon g(t_j)| = |a_j x^{m_j} + 1 - h(t_j + x) + \varepsilon(g(t_j) - g(t_j + x))| \leq C(|x|^{m_j+1} + \varepsilon|x|) \leq C'\varepsilon^{1+1/m_j}.$$

where C, C' are independent of ε. It follows that the roots in each disk are of the form $t_j + \rho_{j,p}\varepsilon^{1/m_j} + o(\varepsilon^{1/m_j})$ where $\rho_{j,p}$, for $p = 1, \ldots, m_j$ are solutions of

$$\rho_{j,p}^{m_j} = -\frac{g(t_j)}{a_j}.$$

Since $\frac{g(t_j)}{a_j}$ for all j are not real, $t_j + \rho_{j,p}\varepsilon^{1/m_j} + o(\varepsilon^{1/m_j})$ for all j are not real for sufficiently small ε and the statement follows. ■

Proof of theorem 4. Pick a policy E^0 and a BGE k^0 corresponding to E^0. By lemma 3 for any $\omega \in \mathbb{R}$,

$$\widehat{\mathrm{k}^K_{E^0}}(\omega) = \frac{f'(k^0) - \widehat{\mathrm{k}^C_{E^0}}(\omega)}{R - \mathrm{i}\omega} \tag{49}$$

Pick a $\delta E \in H$ such that $\int_0^1 \delta E_s ds = 0$. Then defining $\xi_s \stackrel{\text{def}}{=} \mathbb{1}_{0\leq s\leq 1}\int_s^1 e^{\mathfrak{r}v}\delta E^0_v dv$, as in lemma 4, we get $\xi_0 = 0$, and so, by lemma 4,

$$\widehat{\delta \mathrm{k}^C_{E^0+\delta E}}(\omega) = -\frac{\Phi(\mathrm{i}\omega - \varkappa)}{\Phi(\mathfrak{r} - \varkappa)} f''(k^0)\widehat{\xi}(\omega) \tag{50}$$

First, if $\widehat{\mathrm{k}^K_{E^0}}(0) = 1$ we can find $\delta E \in Z$, with $\|\delta E\|$ arbitrarily small, and such that $\widehat{\mathrm{k}^K_{E^0+\delta E}}(0) \neq 1$. Indeed, choose δE such that $\int_0^1 \delta E_t dt = \int_0^1 e^{\mathfrak{r}t}\delta E_t dt = 0$ but $\widehat{\xi}(0) = \int_0^1 te^{\mathfrak{r}t}\delta E_t dt \neq 0$. Such δE exists by Lemma 5 applied to the linear functionals

$$l_1(f) = \int_0^1 f(t)dt,\ l_2(f) = \int_0^1 e^{\mathfrak{r}t}f(t)dt,\ m(f) = \int_0^1 te^{\mathfrak{r}t}f(t)dt,$$

and can be chosen to have an arbitrarily small norm. Then $\widehat{\delta \mathrm{k}^C_E}(0) \neq 0$ and hence $\widehat{\mathrm{k}^K_{E^0+\delta E}}(0) \neq 1$. It follows that we can assume that $\widehat{\mathrm{k}^K_{E^0}}(0) \neq 1$, redefining E^0 if needed.

To prove the statement we will use proposition 4. Define $h\colon S \to \mathbb{C}$ as a complex extension of $\widehat{\mathrm{k}_E^K}$ for some strip S as defined in proposition 4. Let also $g\colon S \to \mathbb{C}$ be the complex extension of $\frac{\Phi(\mathrm{i}\omega-\varkappa)}{(R-\mathrm{i}\omega)\Phi(\mathfrak{r}-\varkappa)} f''(k^0)\widehat{\xi}(\omega)$. Pick $E \in Z$, then there is a collection of $\omega_1, \dots, \omega_k \in \mathbb{R}$ such that $h(\omega_j) = 1$ for all $j \in \{1, \dots, k\}$. We can assume that $\omega_j \neq 0$ based on the argument above.

In order to use the proposition, we need to verify that h and g are analytic on S and that the real part of each at infinity converges to zero. In addition, we need to establish that conditions (48) hold.

For the first part notice that h, g are analytic by lemma 6. Their convergence follows from Riemann-Lebesgue theorem (cf. [5, 22.10.13]).

Conditions (48) require the imaginary part of $\frac{1}{a_j}\frac{\Phi(\mathrm{i}\omega_j-\varkappa)}{(R-\mathrm{i}\omega_j)\Phi(\mathfrak{r}-\varkappa)} f''(k^0)\widehat{\xi}(\omega_j)$ to be distinct from zero, where a_j is the first non-zero coefficient of the Taylor expansion of $h-1$ around ω_j. To avoid computing a_j, denote the real part of $\frac{1}{a_j}\frac{\Phi(\mathrm{i}\omega_j-\varkappa)}{(R-\mathrm{i}\omega_j)\Phi(\mathfrak{r}-\varkappa)} f''(k^0)$ by p and its imaginary part by q. Recall that for $\omega \neq 0$

$$\widehat{\xi}(\omega) = \int_{-\infty}^{\infty} e^{\mathrm{i}\omega t}\xi(t)dt = \int_0^1 \frac{e^{\mathrm{i}\omega t}}{\mathrm{i}\omega} e^{\mathfrak{r}t}\delta E_t dt$$

Hence for $\omega \neq 0$ the imaginary part of

$$\widehat{\xi}(\omega) = \frac{p+iq}{\mathrm{i}\omega}\int_0^1 e^{\mathrm{i}\omega t}e^{\mathfrak{r}t}\delta E_t dt = \Big(\frac{q}{\omega} - \frac{ip}{\omega}\Big)\int_0^1 [\sin(\omega t) + i\cos(\omega t)]e^{\mathfrak{r}t}\delta E_t dt \tag{51}$$

is

$$\int_0^1 \Big(-\frac{p}{\omega}\sin(\omega t) + \frac{q}{\omega}\cos(\omega t)\Big)e^{\mathfrak{r}t}\delta E_t dt \tag{52}$$

For every j define a linear functional $F_j : \delta E \mapsto \frac{1}{\omega_j}\int_0^1 (-p\sin(\omega_j t) + q\cos(\omega_j t))e^{\mathfrak{r}t}\delta E_t dt$. Consider also, as before

$$l_1(f) = \int_0^1 f(t)dt,\ l_2(f) = \int_0^1 e^{\mathfrak{r}t}f(t)dt.$$

Lemma 5 implies that none of F_j is identically 0 on the subspace

$$H_0 := \{f \in H \mid l_1(f) = l_2(f) = 0\}.$$

Hence the set $\{f \in H_0 \mid F_j(f) \neq 0 \text{ for } j = 1, 2, \dots, k\}$ is non-empty; moreover it is dense in H_0 and δE can be picked from this set. ∎

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