Social objectives in general equilibrium

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Abstract

I consider an exchange economy in which each agent's preferences are given by $U_i = u_i + \theta F$, where u_i is a standard utility function, F is a *social objective function* and θ is the weight F receives. Both F and θ are common to all individuals. I show that F's equilibrium value may be a decreasing function of θ. I also show that if F is a social welfare function whose argument are the u_i's, then the economy's equilibria are independent of θ.

Key Words: General equilibrium; Consumption externalities; Other-regarding preferences; Social objectives.
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1 Introduction

Do market outcomes become more egalitarian when society's members develop a taste for equality? More generally, if the preferences of market participants "move in a certain direction," will the market outcome move accordingly? In this paper I address this question in the context of an exchange economy.

Consider an exchange economy in which the preferences of each individual i are given by:

$$U_i = u_i + \theta F, \tag{1}$$

where u_i is a utility function that depends only on i's consumption, F is some function that depends on everything that goes on in the economy (i.e., a function of the entire allocation), and $\theta \in \mathbb{R}$ is a parameter. The function F and the parameter θ are common to all individuals. F is called the *social objective*. The case where the social objective receives no weight—i.e., $\theta = 0$—corresponds to the classical model, in which the consumers are selfish agents who care only about their own consumption.

Consider a set of possible values for the parameter θ, Θ. Suppose, further, that the economy has a unique competitive equilibrium (w.r.t the utilities $\{U_i\}$) for each $\theta \in \Theta$. Denote this equilibrium by $e(\theta)$ and the value of the social objective in this equilibrium by $F(e(\theta))$. Is $F(e(\theta))$ increasing (or, at a minimum, non-decreasing) over Θ?

This paper's first result is an example that shows that both type of relationship are possible: $F(e(\theta))$ can increase and can decrease in θ. In particular, the perverse case, in which placing a higher weight on the social objective by each individual leads to a decrease in the objective's equilibrium-value, is possible. The example is simple: it is a 2×2 economy in which the u_i's are standard utility functions: one is the minimum function and the other is a linear function.

The second result is that if F is isomorphic to a *social welfare function*—namely, $F(x) = W(u(x))$ where x is the allocation, $u(x)$ is the vector of the individual utilities under this allocation and W is some nondecreasing function of the individual utilities—then the set of the economy's competitive equilibria is independent of θ. Moreover, the set of equilibria is independent of the specific F, as long as it is welfarist in the above sense. That is, if F and $\tilde{F}$ are two welfarist social objectives, then their associated economies have the same equilibria. In particular, given a fixed welfarist F, its equilibrium-value is independent of θ.

The rest of the paper is organized as follows. Section 2 contains definitions. Section 3 describes the 2×2-economy example. Section 4 is dedicated to the *welfarism irrelevance* result. Section 5 concludes.

2 Definitions

Consider an exchange economy with L goods and n agents (or consumers), where both L and n are finite. Agent i has an initial endowment $\omega^i \in \mathbb{R}^L_{++}$. His preferences are given by a function U_i that obeys (1), for some social objective F and parameter $\theta \in \mathbb{R}$. The utilities $\{u_i\}$ are called *internal utilities*. The social objective is a real-valued function on $\mathbb{R}^{Ln}_+$. It is *welfarist* if for every specification of n consumption bundles for the n agents, $x = (x^1, \cdots, x^n)$, the following holds: $F(x) = W(u_1(x^1), \cdots, u_n(x^n))$, where the u_i's are the internal utilities and $W \colon \mathbb{R}^n \to \mathbb{R}$ is some non-decreasing function; that is, W is a *social welfare function*. This is a specific (i.e., restrictive) form of welfarism; I will comment on that in Section 4 below.

A *Walrasian*, or *competitive* equilibrium (equilibrium, for short), is defined as usual: it is a pair $(x, p) \in \mathbb{R}^{Ln}_+ \times \mathbb{R}^L$, where x is an allocation and p is a price vector, such that each x_i maximizes U_i on the budget set $B_i(p) \equiv \{y \in \mathbb{R}^L_+ : \sum p_l y_l \leq \sum p_l \omega^i_l\}$ and $\sum_{i=1}^n x^i_l = \sum_{i=1}^n \omega^i_l$ for all $l = 1, \cdots, L$. In order to emphasize the role of θ and F, the equilibria set of the exchange economy $E = [\{\omega^i\}; \{u_i\}; F; \theta]$ is denoted by $\mathcal{E}(E|\theta, F)$. In the specific case where $\theta = 0$ the objective F can be ignored, and I write $\mathcal{E}(E)$ for the set of equilibria.

In the case of a 2-good economy, I use X to denote the first good and Y to denote the second good. The economy is called 2×2 if there are two goods and two agents. In a 2×2 economy I denote the consumption bundles of the two agents by (x_1, y_1) and (x_2, y_2).

Given a set A, ∂A denotes the Pareto-boundary of A. Given a vector v, v_{-i} denotes the vector that one obtains by deleting v's i-th coordinate. Finally, equilibrium-quantities are marked with $*$'s. For example, (x_1^*, y_1^*) is agent 1's equilibrium bundle in a 2×2 economy.

3 The main example

Consider a 2×2 exchange economy where the total resources are $(1, 1)$, agent 1's endowment is (a, b) and that of agent 2 is $(1 - a, 1 - b)$. The internal utilities are $u_1(x_1, y_1) = \min\{x_1, y_1\}$ and $u_2(x_2, y_2) = x_2 + y_2$. The social objective is $F = -|x_1 - x_2|$. Namely, it is socially desirable to increase equality in the consumption of the first good. The parameter θ belongs to $(0, 1)$.

Proposition 1. *Suppose that $a, b > \frac{1}{2}$. Then:*

1. *If $a > b$ then the above economy admits a unique equilibrium. The equilibrium prices and allocation are differentiable functions of θ. The equilibrium-value of the social objective is strictly decreasing in θ.*

2. *If $a < b$ then the above economy admits a unique equilibrium. The equilibrium prices and allocation are differentiable functions of θ. The equilibrium-value of the social objective is strictly increasing in θ.*

Before turning to the proof of this result, let us look at the intuition behind it. The condition $a, b > \frac{1}{2}$ says that agent 1 is richer than agent 2, which, in turn, guarantees that $x_1^* > x_2^*$. Therefore, in equilibrium $F = -(x_1^* - x_2^*)$. If $a > b$ agent 1 trades some of his X's in exchange for some of 2's Y's. Let the equilibrium prices be, wlog,

$(p_X, p_Y) = (p, 1)$.[1] The linearity of 2's utility function dictates that $p = 1 + \theta$. Thus, as θ increases X is becoming more expensive, and as a result agent 2 will buy less of it. Thus, x_2^* decreases and x_1^* increases, implying a decrease in F's value. If $a < b$, the converse holds.

Lemma 1. *If $a, b > \frac{1}{2}$ then there does not exist an equilibrium in which $x_1^* = x_2^*$.*

Proof. Assume by contradiction that an equilibrium with $x_1^* = x_2^*$ exists. Clearly, the budget constraint of agent 2 binds. That of agent 1 also binds; otherwise he would be able to increase his utility by $(1-\theta)\epsilon$, by increasing his consumption of both goods by $\epsilon > 0$. The budget constrains are:

$$px_1^* + y_1^* = pa + b,$$

and

$$px_2^* + y_2^* = p(1-a) + 1 - b.$$

By assumption $x_1^* = x_2^*$, hence $x_1^* = x_2^* = \frac{1}{2}$. Additionally, I argue that $x_1^* = y_1^* = \frac{1}{2}$. To see this, note that $x_1^* < y_1^*$ is impossible since in this case agent 1 could decrease his Y-consumption by ϵ and increase his X-consumption by $\frac{\epsilon}{p}$, which would increase his utility by $\frac{\epsilon}{p}(1-\theta) > 0$. If, on the other hand, $y_1^* < x_1^* = \frac{1}{2}$, then it follows from the budget constraints that $p(1-a) + 1 - b > pa + b$.[2] That is, $1 - 2b > p(2a-1)$. However, this inequality is impossible because $p > 0$ and $a, b > \frac{1}{2}$. Therefore, as argued, $x_1^* = y_1^* = \frac{1}{2}$. Therefore, it follows from agent 1's budget constraint that $\frac{1}{2}(1+p) = pa + b$, or $p(2a-1) = 1 - 2b$, which is impossible because $p > 0$ and $a, b > \frac{1}{2}$. □

[1]Both prices must be strictly positive; otherwise, agent 2 will have infinite demand. Therefore, the normalization $(p, 1)$, where $p > 0$, is wlog. For the rest of this Section I adopt this normalization.

[2]Namely, the RHS of 2's budget constraint is strictly greater than the RHS of 1's budget constraint. The reason is market clearing in the Y-market: $y_1^* < \frac{1}{2}$ implies $y_2^* > \frac{1}{2}$.

Lemma 2. *If $a, b > \frac{1}{2}$ then there does not exist an equilibrium in which $x_2^* > x_1^*$.*

Proof. Assume by contradiction that an equilibrium as above exists. In particular, $x_1^* < \frac{1}{2}$. By the argument from the proof of the previous lemma, agent 1's budget constraint binds: $px_1^* + y_1^* = pa + b$. Also, $y_1^* > x_1^*$ is impossible, as in this case agent 1 could decrease his Y-consumption, increase his X-consumption, and consequently increase his utility. Therefore, since $y_1^* \leq x_1^* < \frac{1}{2}$, it follows from agent 1's budget constraint that:

$$\frac{pa + b}{1 + p} < \frac{1}{2}. \tag{2}$$

Now look at agent 2. He maximizes $x_2 + y_2 - \theta x_2 + \theta x_1$. Since his X-consumption is strictly positive,[3] $p \leq 1 - \theta$. Also, $p < 1 - \theta$ is impossible as in this case agent 2 would consume only X, which implies $y_2^* = 0$ and $y_1^* = 1$—in contradiction to $y_1^* \leq x_1^* < \frac{1}{2}$. Therefore $p = 1 - \theta$. Substituting this into (2) gives:

$$\frac{(1 - \theta)a + b}{2 - \theta} < \frac{1}{2},$$

or

$$2a(1 - \theta) + 2b < 2 - \theta.$$

Since $a > \frac{1}{2}$ and $\theta \in (0, 1)$:

$$1 - \theta + 2b < 2 - \theta,$$

and so $2b < 1$, in contradiction to $b > \frac{1}{2}$. □

Lemma 3. *If $a, b > \frac{1}{2}$ then there exists a unique equilibrium. In this equilibrium $p = 1 + \theta$ and $x_1^* = \frac{(1+\theta)a+b}{2+\theta} > x_2^*$.*

[3] There is market clearing in the X-market and $x_1^* < \frac{1}{2}$.

Proof. By Lemmas 1 and 2, if there is equilibrium then it is such that $x_1^* > x_2^*$. In particular, agent 2 maximizes $x_2(1+\theta)+y_2-\theta x_1^*$. In this equilibrium $x_1^* = y_1^*$: if $x_1^* > y_1^*$ then agent 1 could increase his utility by decreasing his X-consumption and increasing his Y-consumption, and if $x_1^* < y_1^*$ then an ϵ-decrease of Y brings an increase of $\frac{\epsilon}{p}$ units of X, which increases utility by $\frac{\epsilon}{p}(1-\theta)$. Therefore the allocation is a point on the Edgeworth box diagonal. Obviously, in equilibrium $(x_1^*, y_1^*) \notin \{(0,0),(1,1)\}$; therefore, the equilibrium allocation is in the interior of the box. Therefore, due to the linearity of agent 2's utility in X, it follows that $p = 1+\theta$. Substituting this into 1's budget constraint gives $x_1^* = \frac{(1+\theta)a+b}{2+\theta}$. It is easily verified that each agent's consumption bundle is optimal for him. □

Equipped with the lemmas, we can turn to the proof of the proposition.

Proof of Proposition 1: By the lemmas, $x_1^* = \frac{(1+\theta)a+b}{2+\theta} > x_2^*$. Therefore, $\Delta \equiv -F = x_1^* - x_2^* = x_1^* - [1-x_1^*] = 2x_1^* - 1$. Therefore, $\text{sign}\frac{\partial \Delta}{\partial \theta} = \text{sign}\frac{\partial x_1^*}{\partial \theta}$. Finally, $\frac{\partial x_1^*}{\partial \theta} = \frac{a-b}{(2+\theta)^2}$, which is strictly positive if $a > b$ and strictly negative if $a < b$. □

4 Welfarism irrelevance

Proposition 2. *Let E be an exchange economy, let $\theta \in \mathbb{R}$, and let F be a welfarist social objective. Then $\mathcal{E}(E) = \mathcal{E}(E|\theta, F)$.*

Proof. Consider an exchange economy, E, the internal utilities of which are $\{u_i\}$. Let $\theta \in \mathbb{R}$ and let F be a social objective. Let $(x,p) \in \mathcal{E}(E)$. Each x^i maximizes $u_i(.)$ over $B_i(p)$. Therefore, each x_i maximizes the following expression over $B_i(p)$:

$$u_i(.) + \theta W(u_1(x^1), \cdots, u_{i-1}(x^{i-1}), u_i(.), u_{i+1}(x^{i+1}), \cdots, u_n(x^n)). \tag{3}$$

Therefore, $(x,p) \in \mathcal{E}(E|\theta, F)$. Conversely, let $(x,p) \in \mathcal{E}(E|\theta, F)$. Then each x_i maximizes expression (3) over $B_i(p)$. Therefore, each x^i maximizes $u_i(.)$ over $B_i(p)$. Therefore, $(x,p) \in \mathcal{E}(E)$. □

Proposition 2 can be strengthened. First, note that the set of equilibria is not only independent of θ, but, in fact, is independent of F, as long as it is welfarist. That is, if F and $\tilde{F}$ are two welfarist social objectives, then their associated economies have the same equilibria. The proof of this result is essentially identical to the one given above and it is therefore omitted.

Also, the notion of welfarism can be broadened, as follows: instead of assuming that F is common to all individuals and $F(x) = W(u(x))$, where W is weakly increasing in all n-dimensions, one can alternatively assume that i's objective is $u_i + \theta F_i$ with $F_i(x) = W_i(u(x))$, and where W_i is only increasing in u_i, not necessarily in $\{u_j\}_{j \neq i}$. A simple example of preferences of this sort is given by the following functional form, that dates back to Edgeworth (1881):

$$W_i(u_1, \cdots, u_n) = u_i + \frac{\beta_i}{n-1} \sum_{j \neq i} u_j.$$

The parameter β_i is allowed to be negative, meaning that agent i is spiteful. Another example is given by the following preferences, due to Bolton and Ockenfels (2000):

$$W_i(u_1, \cdots, u_n) = u_i - \beta_i |u_i - \frac{\sum_j u_j}{n}|,$$

where $\beta_i \in [0, 1)$. Functional forms like the abovementioned two can serve to model what is known as "keeping up with the Joneses." Namely, that relative standing in the society affects well-being. With i's preferences given by $u_i + \theta F_i$ as above, a result analogous to Proposition 2 obtains, with the proof being essentially identical to that of Proposition 2.

Preferences of the form $u_i + \theta F$ are a particular kind of *other-regarding preferences* (ORP). Dufwenberg et al. (2011) study ORP in a general equilibrium framework and find a necessary and sufficient condition on preferences for the coincidence of ORP-behavior with selfish behavior. Consequently, under their condition there is coincidence of the set of equilibria of two economics: an ORP economy and the classical economy that is naturally associated with the ORP economy.[4] Their condition, called *separability*, states that an agent's preference over his part of the allocation is independent of what the other agents receive. That is, the ranking between any two bundles, x_i and x'_i, is independent of x_{-i} and x'_{-i}. Preferences of the form $u_i + \theta F$, where F is welfarist, are generally not separable. Proposition 2 is logically incomparable to the result of Dufwenberg et al. (2011): the later provides a necessary and sufficient condition for the irrelevance of ORP to individual behavior, whereas the former only describes a sufficient condition for the ORP's irrelevance for the equilibria set. Both results, however, share a common flavor: they are *irrelevance results*, demonstrating that certain enrichments of the classical general equilibrium model do not have observable implications.

The present model can be modified as to demonstrate a further irrelevance result. Consider the case where F depends not only the economy's allocation, but, additionally, on prices. Specifically, consider the case where F in (1) is replaced by G, where $G\colon \mathbb{R}^n_+ \to \mathbb{R}$ is a function of the n individual expenditures: $G = G(p \cdot x^1, \cdots, p \cdot x^n)$, where p is the price vector and x is the allocation. Let $E_i = p \cdot x_i$ denote agent i's expenditure. If each agent exhausts his entire budget, then the equilibria of the corresponding economy are independent of G, since $E_i = p \cdot \omega_i$ in every equilibrium.[5]

5 Conclusion

I have considered a departure from the classical exchange economy model, by introducing a common component to the individual utility functions: θF. F is a function of the entire allocation—the *social objective*—and θ is the weight it receives. It is possible for F's equilibrium-value to be a decreasing function of θ. In this sense,

[4] In the present model "naturally associated" means $\theta = 0$. In Dufwenberg et al. (2011) this has a more general meaning.

[5] Note that, no matter what G is, agent i's problem is to maximize $u_i(x^i) + \theta G(p \cdot x^i, E_{-i})$ over $x^i \in \partial B_i(P)$.

the market mechanism may fail to reflect the preferences of the market participants. This pathology has a similar flavor to a one documented in Dow and Werlang (1988), where it is shown that the utility of a market's representative consumer may increase if the utility of each actual market participant decreases.[6]

A closely-related issue was studied by Milchtaich (2012) in the context of symmetric games and population games. He showed that in these strategic models, if one adds a social payoff component to the individual utility functions, then the social payoff's equilibrium value may be non-monotonic in the social payoff's coefficient. That is, the analog of $F(e(\theta))$ in the strategic model can decrease in θ. However, it can only decrease if the equilibrium strategies are, in a specific sense, unstable; if they are stable, then $F(e(\theta))$ can only increase or remain unchanged as θ increases. The stability concept does not have an analog in the market model.

If F's argument is the vector of individual utilities, then F has no impact on the set of the economy's equilibria. In this sense, the classical market model is robust to perturbations: adding a common component to the individuals' preferences in the form of a welfarist function does not affect the equilibria set. This robustness result is robust (pardon the pun): it continues to hold if F is replaced by individual welfare functions that only increase in own-utility, and also when F depends on the individuals' expenditures.

References

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[2] Dow, J. and da Costa Werlang, S.R., 1988. The consistency of welfare judgments with a representative consumer. *Journal of Economic Theory*, **44**, 269-280.

[6]A representative consumer is a fictitious individual whose utility function gives rise to a demand correspondence that coincides with the market's aggregate demand.

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