# CONTEMPLATION VS. INTUITION. A REINFORCEMENT LEARNING PERSPECTIVE.

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Abstract. In a search for a positive model of decision-making with observable primitives, we rely on the burgeoning literature in cognitive neuroscience to construct a three-element machine (agent). Its control unit initiates either impulsive or cognitive element to solve a problem in a stationary Markov environment, the element "chosen" depends on whether the problem is mundane or novel, memory of past successes and the strength of inhibition.

Our predictions are based on a stationary asymptotic distribution of the memory, which, depending on the parameters, can generate different "characters", e.g., an *uptight dimwit*, who could succeed more often with less inhibition, as well as a *relaxed wiseguy*, who could gain more with a stronger inhibition of impulsive (intuitive) responses.

As one would expect, stronger inhibition and lower cognitive costs increase the frequency of decisions made by the cognitive element. More surprisingly, increasing the "carrot" and reducing the "stick" (being in a more supportive environment) enhances contemplative decisions (made by the cognitive unit) for an alert agent, i.e., the one who identifies novel problems frequently enough.

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## 1. An anecdote to start

The reader can probably confirm that many academics tend to raise their children in a "supportive" environment, and the grown offsprings rather often decide to become academics themselves. This should be surprising in view of the classical mechanism design literature, where payoffs can be "normalized" so that only the gap between the best and the worst payoff can affect behaviour, their shift being of no consequence. If explaining this phenomenon is sufficient for the motivation, please proceed to section 3, describing the model. Alternatively, please go to the next section containing "deeper reasons" for the chosen approach.

## 2. Conceptual motivation and related research

Our goal is to describe a technology, the technology of decision making. Why? Even the most ardent supporters of rational decision-making may be disappointed with their own decisions that are made hastily, without much thought. Indeed, how often do we know what *should be* done, but act differently? More importantly, the observed individual behaviour departs from rationality (even most broadly defined as in Gilboa and Schmeidler (2001)), and this is a well-established fact.[1] We do not intend to enter a debate about economic significance of this departure, however, we hope the reader will agree that it is useful to construct a model that could help us understand when we are more likely to buy what we do not need or trust unreliable politicians, at least for building meaningful predictions.

Also, we are not even attempting to offer a glimpse into the recent literature that uses an axiomatic approach to incorporate various forms of "irrational" behaviour, though, no doubt, it produced most elegant and insightful results.[2] Although it might be due to our ignorance, but we were unable to find a convincing tractable unified framework using that route. And by *convincing and tractable* we mean a model that can be explained to undergraduate students. The main problem, most likely, is the lack of primitives that can be directly observed: neither preferences nor beliefs fall in this category. To stress, in no way are we disputing the importance of the traditional primitives for clear and logical analysis of how decisions *should* be made. Our goal is to use "natural laws" to describe how decisions *are* being made.

[1]Cf. the work of Daniel Kahneman and Amos Tversky, or of Malcolm Gladwell for example.

[2]A.o., it includes research in neuroeconomics, see Glimcher and Fehr (2013) for an overview.

Here is our approach in a nutshell. "Nature" has endowed us with a machine which can encounter problems from the environment and produce solutions (with some probability of success) as an output. The machine can store/encode past experiences. Partly, the mechanism is given by nature, partly, it is affected by the environment ("nurture"). To be more precise we rely on selected findings from biology and cognitive psychology and sketch briefly the closely related literature in economics.[3]

### 2.1. **Stylized facts.**

2.1.1. *Thinking requires resources.* For a recent study reporting differential (by task difficulty and area of the brain) glucose consumption see McNay et al. (2000).

The idea that cognitive activity requires resources that have an alternative cost, of course, has been used even in economics, see for example, Woodford (2012) and Sims et al. (2010), who provide a rationale for various degrees of sophistication in solving problems.

In addition, there is some evidence that fast and easy actions trigger immediate rewards,[4] and that can be considered as an alternative cost of thinking.

2.1.2. *Existence of different decision-making units.* To start, let us mention the pioneering work of Ferrier (1876) who distinguished between impulsive and deliberate modes of decision making:

> If the centres of inhibition, and thereby the faculty of attention, are weak, or present impulses unusually strong, volition is impulsive rather than deliberate. (Ferrier, 1876, p. 287).

As for "state of the art" in neurobiology on the subject, cf. the review by Bari and Robbins (2013), who stress the hierarchical nature of the process and discuss the role of the control center:

> In general, executive control is thought to operate in a hierarchical manner with the PFC [prefrontal cortex] having a leading role over lower-level structures...(Bari and Robbins, 2013).

The 'machine' we model has three basic components: a *cognitive* unit, an *impulsive* unit and a *control* unit. Thus, we distinguish between

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[3] For an overview in the latter field, see, for example, Corr and Matthews (2009, parts IV, V), in particular, uncovering physiological explanations for various "personality traits" and their effect on behaviour.

[4] See Chung et al. (2015).

pure "problem solving" tasks that might involve activating working and long-term memory, done by our *cognitive unit*, and executive control function (the *control* unit), which is responsible for inhibiting fast "instinctive" responses (by the *impulsive* unit), a.o., in go/no-go tasks. The related neurobiological research is burgeoning,[5] although there are still a lot of open questions related to understanding executive functions of the brain and their hierarchy. Nevertheless, being able to identify the part of the brain responsible for a particular impulse inhibition allows not only to detect its activation during various tasks, but also to carefully study the relevant triggers which alter the resulting behaviour (Spieser et al., 2015). With the triggers weakening or enhancing the control unit, the picture of the brain operation emerging from this literature is, indeed, that of a 'machine'.

Not surprisingly, the distinction between "fast" and "slow" decision-making has found its way into economics literature, probably due to the recent book by Kahneman (2011). In fact, a multi-level decision process appeared already in the celebrated chain-store paradox by Selten (1978). The recent two-mode decision making models, Alos Ferrer (2013); Cerigioni (2015), still retain, however, at least in part, the classical primitives (preferences), which we do not adopt here.

2.1.3. *Interpersonal differences in inhibition control.* These differences might be responsible for individual traits (some forms of impulsiveness and even risk-taking),[6] which affect one's behaviour sometimes against one's own will.[7] In part, we will attempt to identify the strength of the inhibition control (parameter $h$ in our model) from the observed behaviour, i.e., activation of cognitive or impulsive unit to solve problems, and that we take as an "observed variable".

2.1.4. *Fast and easy actions trigger immediate rewards.* The three decision-making units are analysed in the context-dependent choice environment in the recent study by Chung et al. (2015). Among other findings is the observation that fast decision-making bears a pleasurable reward associated with the process itself, rather than with the eventual outcome.

2.1.5. *Experience matters.* Past experiences are encoded in memory. Here we focus on "emotional" memory of past successes and failures,

[5]Cf. Smith and Jonides (1999); Shomstein (2012); Bari and Robbins (2013); Ding et al. (2014).

[6]Although some of these connections are subject to additional scrutiny, Brown et al. (2015).

[7]Cf. von Hippel and Dunlop (2005), see also an overview in Kahneman (2011).

see, e.g., Fujiwara and Markowitsch (2006) for a summary of the relevant studies.

2.1.6. *Novel situations might activate cognitive functions.* For partial support see Corbetta et al. (2008): the nature of the mechanism is rather involved.

## 3. The model

3.1. **The decision-making process.** As mentioned in the introduction, the decision-making process has three components (units): impulsive, cognitive, and control. Cognitive (or contemplative) unit expands resources during its search for an appropriate action. Impulsive unit is fast, provides an answer based on "intuition" and requires minuscule resources to operate. Control unit can stop the impulsive one from triggering the action and shift the decision to the cognitive unit. Exercising control has a cost ($\kappa > 0$): first, the foregone pleasure from the impulsive decision; second, the added reaction time; and third, the energy spent by operating the cognitive unit itself.

3.2. **The environment.** Here we abstract away from the process of finding a solution to every problem, taking it as a black box, acknowledging however, that many interesting issues are omitted by doing so. We start with a rather informal description of the environment.

There are 2 possible problems in the environment faced by our decision maker (DM): simple and difficult (novel).

Each period DM is confronted with a simple problem with probability $0 < w < 1$ (and the difficult one shows up with the complementary probability).

*Interpretation.* $w$ measures how challenging is the environment surrounding the DM. In general one might want to have $w$ potentially changing with experience, as in the same environment difficult problems become simple with skill acquisition, but our DM constantly faces new challenges with probability $1 - w$.

A simple problem is similar to the problems encountered in the past, and the impulsive unit identifies the right solution with probability $0 < p < 1$ each time and cognitive unit with probability $0 < a \leq 1$.

*Interpretation.* $p < 1$ reflects inability of the impulsive unit to process all the relevant information in a short time, since it is using a rule of thumb that is either learnt from the past or is programmed ahead of time via evolutionary pressures. $a \leq 1$ embeds the possibility of failing to find solutions for simple problems, even when thinking is involved.

Control unit 'decides' whether cognitive or impulsive unit will be activated based on the history of past successes and the type of task.

A difficult or a novel situation requires introspection and is identified as such (by the control) with probability $0 < z < 1$. If it is identified as difficult, the control activates the cognitive unit, which succeeds in finding the best solution with probability $0 < g < 1$. If control unit fails to identify the problem as difficult, then it uses the same algorithm as in case of a simple problem. If impulsive unit is activated, the success rate, $b$, is low: $0 < b < 1$.

*Interpretation.* Some new situations are obviously different, others have more hidden signs, thus are harder to detect, so $z < 1$. There is no clear way to teach to identify new situations (recall "Clever Hans" from Brothers Grimm (1826)), so $z$ probably reflects an in-born quality. Missing relevant information in each case or cognitive limitations in general are reflected in assuming $g < 1$.

Note that we did not assume that cognitive unit always outperforms the impulsive one in finding the right solutions to various problems: in fact, there are many decisions for which the opposite might be true (e.g., choosing how to breathe and walk). One might conjecture that if the cognitive unit is less successful and more costly, it should not be used at all, unless called for by the novelty of the problem (which we assumed), but, surprisingly, this conjecture might be wrong.

3.3. **Memory and control unit "decisions".** Memory at time $t$ is a pair of real numbers, $I_t, C_t$:

- $I_t$ indicates (relative) performance of the impulsive unit, i.e., observed frequency of its successes;
- $C_t$, similarly, reflects the performance of the cognitive component less the cost, $\kappa > 0$, expanded from its activation.

Given the inhibition parameter $h$, if a simple problem is encountered control unit activates

| | |
|---|---|
| cognitive unit if | $I_t < h + C_t$ |
| impulsive unit if | $I_t > h + C_t$ |
| either one | otherwise |

If a novel/difficult problem is encountered, the algorithm is the same, unless the novelty is detected and then the cognitive unit is used, independently of the memory, $I_t$ and $C_t$.

*Interpretation.* Following section 2.1.3 in the introduction, we view the inhibition parameter, $h$, mainly as an in-born trait, although one might entertain a view that at least partially, the strength of inhibition can be modified by the appropriate training. To stress, again, although this parameter can not be readily "measured", which unit is activated — impulsive or cognitive — can, by and large, be observed.

An active unit triggers an action that generates a reward if successful, and incurs a loss in case of failure. $r$ and $l$ stand for reward and loss, correspondingly, each depending on whether the problem is simple, $m$, or difficult, $f$. This "instantaneous" payoff reinforces each one of the units, i.e., is used to update the indicators of the relative performance.

*Interpretation.* We assume that the payoffs include the "immediate" gratification from solving the problem correctly in the first case and, similarly, discouragement in case of failure. These parameters can be thought of as reflecting the environment (nurture): more approval for success (from peers, parents, others) translates into higher $r$ and more discouragement, heavier punishment yields higher $l$.

Of course, these rewards and losses are also, in part, a reflection of the subject's own attitude: how critical the DM is of his failures and how proud he is of success. We could potentially connect it to a variable of choice: one can decide to change the attitude and increase own $r$ or $l$ via training/self-conviction. The question is then: what will the effect be?

3.4. **Formal description of the model.** As follows from the assumptions, the instantaneous payoff is a random variable. Its realizations are pairs of real numbers, indicating the reward for the cognitive and

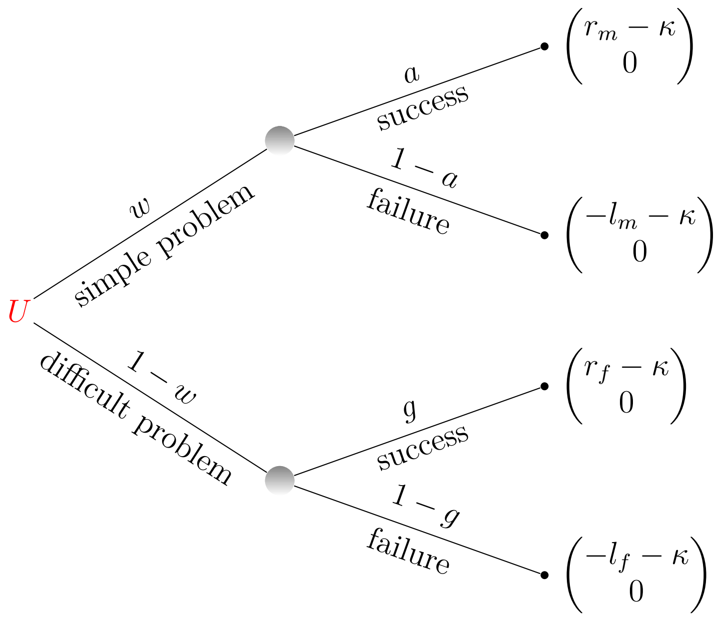


FIGURE 3.1. Random payoff, $U$, in the cognitive regime. Notice that the impulsive unit is not rewarded, so the second component of the realized payoff in $U$ is zero.

impulsive units correspondingly. Its distribution is governed by the independent random events: whether the problem is simple or difficult, and whether the unit solving the problem is successful or not.

The distribution over these realizations depends on the sign of $I_t - (C_t + h)$.

If the sign is negative, so that $I_t < C_t + h$, we say the agent is in the *cognitive regime*: only the cognitive unit is active and can generate non-zero instantaneous rewards.

Denote the random instantaneous reward in cognitive regime by $U$, as described in figure 3.1.

The updated memory is a weighted average of an instantaneous net reward and the old indicator. The relative weight is $0 \le \beta \le 1$. So, when in cognitive regime,

$$\begin{pmatrix} I_{t+1} \\ C_{t+1} \end{pmatrix} = \begin{pmatrix} 1-\beta & 0 \\ 0 & 1-\beta \end{pmatrix} U + \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} I_t \\ C_t \end{pmatrix} \tag{1}$$

Now, if $I_t > C_t + h$, the agent is in the *impulsive regime*: impulsive unit is activated unless a difficult problem is detected. Recall, the problem is simple with probability $w$, otherwise, when the problem is difficult and it is detected (with probability $z$), cognitive unit is always

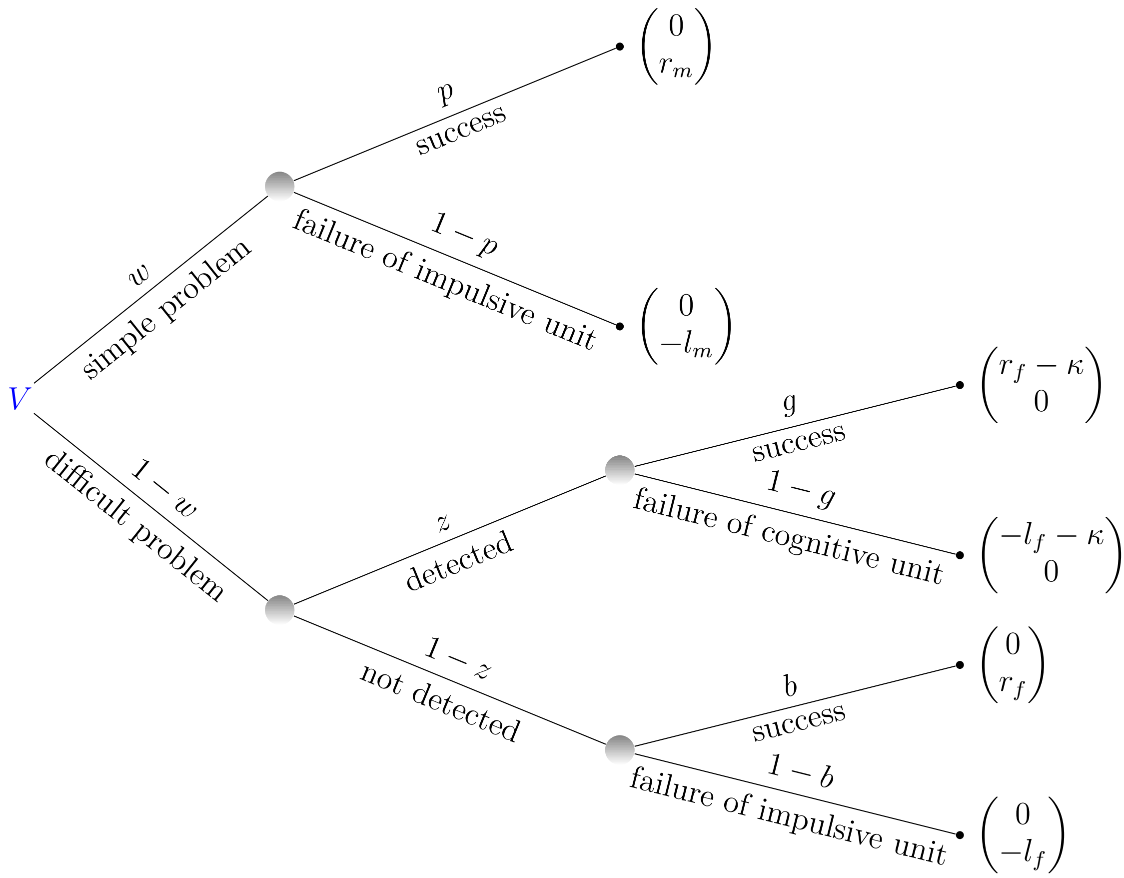


FIGURE 3.2. Random payoff, $V$, in the impulsive regime. Note that only when the cognitive unit is activated, the payoff is reduced by $\kappa$.

activated. The instantaneous reward is a random variable $V$, described in figure 3.4. The memory is updated according to the same algorithm as in the previous case:

$$\begin{pmatrix} I_{t+1} \\ C_{t+1} \end{pmatrix} = \begin{pmatrix} 1-\beta & 0 \\ 0 & 1-\beta \end{pmatrix} V + \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} I_t \\ C_t \end{pmatrix} \tag{2}$$

In case of equality, $I_t = C_t + h$, any convex combination of the two update rules can be used:

$$\begin{pmatrix} I_{t+1} \\ C_{t+1} \end{pmatrix} = \begin{pmatrix} 1-\beta & 0 \\ 0 & 1-\beta \end{pmatrix} (qV + (1-q)U) + \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} I_t \\ C_t \end{pmatrix}, \quad q \in [0,1] \tag{3}$$

To sum up, the memory update can be described using the indicator functions, $\mathbb{1}$:

$$\begin{pmatrix} I_{t+1} \\ C_{t+1} \end{pmatrix} = \begin{pmatrix} 1-\beta & 0 \\ 0 & 1-\beta \end{pmatrix} \Big[ \mathbb{1}_{I_t = C_t + h}(qV + (1-q)U) + \\ + \mathbb{1}_{I_t > C_t + h} V + \mathbb{1}_{I_t < C_t + h} U \Big] + \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} I_t \\ C_t \end{pmatrix}, \quad q \in [0,1] \quad (4)$$

Next, we want to describe the long-term (asymptotic) behaviour of the system:[8] how often the problems are solved correctly, how often each of the modes is being used. And of course, we could then trace the dependence of the long-term behaviour on the parameters of the model: probabilities of success of each unit and the payoffs.

*Notation.* Let

$$i \stackrel{\text{def}}{=} (r_m, -l_m, r_f, -l_f) \in \mathbb{R}^4$$
$$c \stackrel{\text{def}}{=} i - (\kappa, \kappa, \kappa, \kappa)$$

Then for the cognitive regime define:

$$\pi_{c1} \stackrel{\text{def}}{=} \big(wa, w(1-a), (1-w)g, (1-w)(1-g)\big)$$
$$\hat{C} \stackrel{\text{def}}{=} \pi_{c1} \cdot c$$

For the impulsive regime, let

$$\pi_{i2} \stackrel{\text{def}}{=} \big(wp, w(1-p), (1-w)(1-z)b, (1-w)(1-z)(1-b)\big)$$
$$\pi_{c2} \stackrel{\text{def}}{=} z(1-w)\big(0, 0, g, (1-g)\big)$$
$$\bar{I} \stackrel{\text{def}}{=} \pi_{i2} \cdot i, \quad \bar{C} = \pi_{c2} \cdot c$$

## 4. Benchmark: amnesiac ($\beta = 0$) long-term behaviour

To start, let us analyse the simplest case, with only the short-term memory, i.e., when $\beta = 0$. In addition in this section we will also set $a$ to unity implying that the cognitive unit always succeeds in solving simple problems.[9] We will (somewhat loosely) call such an agent an amnesiac, referring, of course, only to the very short "emotional" memory associated with past rewards and successes.

In this case, the process has a finite number of states. The state describes which unit was activated (C or I), whether the problem was simple or difficult (M or F) and whether the unit succeeded in its task (S or L). In other words, the set of states can be identified with a set

[8] It will be later shown that $(I_t, C_t)_{t=0}^{\infty}$ can be represented as a Markov process.

[9] This asymmetry between the two problem-solving units makes the analysis more interesting.

of realizations of payoffs, i.e., possible values of $(C_t, I_t)$. Alternatively, this can be thought of as a set of all the end notes in probability trees $V$, $U$.

List the states as follows: CMS, CML, CFS, CFL, IMS, IML, IFS, IFL.

To construct the probability transition matrix, $\Pi$, first, observe that according to the model the transition probabilities from each of the states can be of two types, either $(\pi_{c1}, 0, 0, 0, 0)$ or $(\pi_{c2}, \pi_{i2})$.

We illustrate the construction for certain "extreme" values of the inhibition parameter, $h$, that will give us also (immediately) the long-run behaviour of the Markov chain.

If $h > r_f$, then even if the impulsive unit succeeded in solving the difficult problem correctly, the agent switches to a cognitive regime, assuming $r_f > r_m > -l_m > -l_f$, the same is true after any activation of the impulsive regime. If $h > l_f + \kappa > \kappa - r_f$, then even after the cognitive unit makes a mistake, it still prevails. Hence if $h > \max\{r_f, l_f + \kappa\}$ then $(\pi_{c1}, 0, 0, 0, 0)$ is the only row that will appear in the transition matrix in this case. Hence this distribution is the long-run distribution over the eight states: since each row sums up to unity and each column consists of identical elements (as all rows are the same), $\Pi^T = \Pi$ for all $T$, and so $\lim_{T\to\infty} \Pi^T = \Pi$.

If, on the other hand, $h < \min\{-l_f, -l_m\}$, then each row of the transition matrix is $(\pi_{c2}, \pi_{i2})$, which, similarly, is also the asymptotic distribution in this case. To sum up,

**Lemma 1.** *If $h > \max\{r_f, l_f + \kappa, r_m, l_m + \kappa\}$, then only cognitive unit is used with positive probability and the distribution over the eight states* CMS, CML, CFS, CFL, IMS, IML, IFS, IFL *is $(\pi_{c1}, 0, 0, 0, 0)$. If $h < \min\{-l_f, -l_m\}$, then the impulsive unit is used unless a difficult problem is detected and the asymptotic distribution is $(\pi_{c2}, \pi_{i2})$.*

Is there an asymptotic distribution in non-extreme cases?

A classical candidate for the asymptotic distribution is the stationary one, i.e., $x$ such that $x\Pi = x$. We will establish existence of such $x$ using Perron-Frobenius theorem, but before the result is established, let us look at an example.

**Example 1.** Assume (1) $r_f > r_m$; (2) $h > \max\{-l_f, -l_m\}$; (3) $r_m - \kappa + h > 0$; (4) $\min\{r_m, l_m + \kappa, l_f + \kappa\} > h$. Under these assumptions the decision is done by the unit that succeeded in the last round and the control is switched to a different unit after experiencing a loss. Then

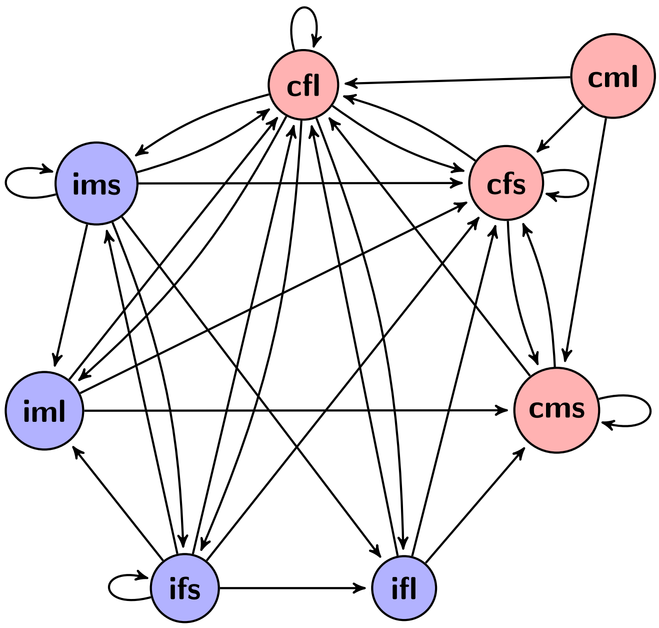


FIGURE 4.1. The Markov chain corresponding to the matrix (5), i.e., assuming $r_m > h > \kappa - r_m$. The nodes represent the states and the arches (edges) correspond to the non-zero transition probabilities. The states colored blue are those where impulsive unit is responsible for solving the problem, and the red states are those where cognitive unit is.

the transition matrix, $\Pi$ is

| | CMS | CML | CFS | CFL | IMS | IML | IFS | IFL |
|---|---|---|---|---|---|---|---|---|
| CMS | $\pi_{c1}$ | | | | 0 | 0 | 0 | 0 |
| CML | $\pi_{c1}$ | | | | 0000 | | | |
| CFS | $\pi_{c1}$ | | | | 0 | 0 | 0 | 0 |
| CFL | $\pi_{c2}$ | | | | $\pi_{i2}$ | | | |
| IMS | $\pi_{c2}$ | | | | $\pi_{i2}$ | | | |
| IML | $\pi_{c1}$ | | | | 0 | 0 | 0 | 0 |
| IFS | $\pi_{c2}$ | | | | $\pi_{i2}$ | | | |
| IFL | $\pi_{c1}$ | | | | 0 | 0 | 0 | 0 |

(5)

$\Pi$ describes a Markov chain, which can be presented as a directed graph, cf. figure 4.1.

It can be verified using figure 4.1 that, excluding CML, there is a path from each state into every other state, so the Markov (sub)chain excluding CML is irreducible. It also implies that from every such state there is a path that ends in the same state and passes through, say, IFS. But then there is also such a loop that has one more transition (edge), as there is a positive probability that the state IFS will be repeated. Hence the chain excluding CML is irreducible and aperiodic, i.e., for $T$ high enough $\Pi^T$ will have all its columns non-zero apart from CML. Irreducibility and aperiodicity together are sufficient for the Markov chain to have a unique stationary distribution by Perron-Frobenius theorem. This analysis is easy to generalize to include the rest of the cases. We will only assume for the rest of this section that $r_f > r_m$ and $l_f > l_m$, since difficult/novel problems typically generate wider-spread gains. Besides, these assumptions make the arguments shorter.

**Proposition 1.** *If $\beta = 0$, then $(C_t, I_t)$ is a finite Markov chain with eight states and it has a stationary (asymptotic) distribution.*

*Proof.* $(C_t, I_t)$ is a finite Markov chain with eight states (listed in lemma 1) by construction.

The existence and uniqueness of the stationary distribution in the extreme cases is by lemma 1. It remains to consider the complementary case: $-l_f < h < \max\{r_f, l_f + \kappa\}$.

First, note that $h > -l_f$ implies that a failure in the impulsive regime (after solving at least a difficult problem) will cause the control unit to switch to the cognitive regime, i.e., the transition probability in state IFL is $(\pi_{c1}, 0, 0, 0, 0)$. So once the agent is in the impulsive mode, there is a positive probability to switch to the cognitive regime. Thus, in the graph of the chain, there is a path from any impulsive state IMS, IML, IFS, IFL to any cognitive state CMS, CFS, CFL (excluding CML).

Assume now that $\max\{r_f, l_f+\kappa\} = r_f$ and $r_f > h > l_f+\kappa$. $h > l_f+\kappa$ implies that the agent will remain in the cognitive regime even after making a mistake, so once in the cognitive regime, he will remain there with probability 1, so there is no path (in the graph of the chain) from a cognitive state to an impulsive state. Thus in this case the subchain consisting of the "cognitive states" CMS, CFS, CFL is recurrent and aperiodic (the latter is because there is a positive probability that any of the three states will be repeated).

Alternatively, $h < l_f + \kappa$, so a failure of the cognitive unit to solve a difficult problem triggers a switch to the impulsive mode, i.e., the CFL row of the transition matrix is $(\pi_{c2}, \pi_{i2})$. Thus, in the graph of the chain there is a path from CFL (and so also from CMS, CFS) to any of the impulsive states IML, IMS, IFL, IFS.

It follows that the seven states comprise an irreducible Markov chain. It remains to show aperiodicity.

If $\kappa - r_f > h > r_f$ then the agent switches from one regime to the other with probability one every period. Thus half of the time will be spent in each regime, so there is a positive probability that a state, say CFL, will be repeated.

If $h < r_f$ then a success in the impulsive regime will keep the agent in that regime for the next round, i.e., the transition probability distribution in the state IFS is $(\pi_{c2}, \pi_{i2})$. Thus, in particular, there is a positive probability that the state IFS will be reached twice consequtively.

It follows that in this case the seven states (all but CML) comprise an irreducible aperiodic chain, hence by the Perron-Frobenius theorem this chain has a stationary distribution. CML can not be reached from any other state (it is transient), hence it is not in the support of that distribution. □

The distribution can be computed explicitly by solving equation $x\Pi = x$ with the relevant transition matrix.[10]

4.1. **Some "comparative statics" for the amnesiac case.** The calculation of the long-run behaviour can be simplified, however, if one re-formulates the problem as a two-state Markov chain with the states being impulsive regime (IR) and cognitive one (CR). Note that the frequency of usage of each regime uniquely determines the frequencies with which each of the units, impulsive and cognitive will be used. These latter frequencies we take as an "observed variable".

Then the transition matrix $P$ for the two regimes is

$$\begin{array}{ccc} & \text{CR} & \text{IR} \\ \text{CR} & \alpha & 1-\alpha \\ \text{IR} & 1-\gamma & \gamma \end{array} \tag{6}$$

where $\alpha$ and $\gamma$, both in $[0, 1]$, are determined by the relative magnitudes of $h$ and the payoffs. For the example discussed above, probability of staying in the cognitive regime is $\alpha = \pi_{c1} \cdot (1, 1, 1, 0)$ and probability of staying in the impulsive regime is $\gamma = \pi_{i2} \cdot (1, 0, 1, 0)$.

If $\alpha = \gamma = 0$, the impulsive regime is followed with probability one by the cognitive regime and vice versa, so the agent spends half of the time in either regime.

[10]Note though that $\Pi - \mathbf{I}$ (with $\mathbf{I}$ being the identity matrix and $\Pi$ any $n \times n$ stochastic matrix) is not invertible: unity is an eigenvalue of $\Pi$ by the Perron-Frobenius theorem and it has the corresponding non-negative eigenvector.

The set of non-trivial non-negative solutions of the system $x(\Pi - \mathbf{I}) = 0$ is a line (there is a single "degree of freedom"), and requiring the sum of elements of the solution ($x$) to be unity uniquely identifies the vector.

The two-state Markov chain is irreducible and is aperiodic unless $\alpha = \gamma = 0$.

So if $\alpha$ and $\gamma$ are not both zero the chain has a stationary distribution $z = (1-q, q)$: $zP = z$, so $q$ solves

$$(1-q)\alpha + q(1-\gamma) = 1-q \tag{7}$$

and so $q = \frac{1-\alpha}{2-\alpha-\gamma}$.

Thus $q$ is decreasing in $\alpha$ and increasing in $\gamma$. Note that $\alpha$ is, for an arbitrary combination of parameters, a product of $\pi_{c1}$ and a vector of four 0-1 indicators of whether the agent stays in the cognitive regime after the cognitive unit encountered each one of the two problems (simple of difficult) and whether it succeeded in each. Thus, $(1,1,1,0)$ stands for the case where the agent stays in the cognitive regime unless the unit failed to solve a difficult problem. Similarly, $\gamma$ is the product of $\pi_{i2}$ and the corresponding four indicators (1 for staying in the impulsive regime and 0 for switching to the cognitive one).

In order to see how $\alpha$ and $\gamma$ change with $h$, let us start with the extreme case $h > \min\{r_f, l_f + \kappa\}$. In this case $\alpha = \pi_{c1} \cdot (1,1,1,1) = 1$ and $\gamma = \pi_{i2} \cdot (0,0,0,0)$. Thus $q = 0$, which is consistent with agent staying in cognitive regime with probability 1. At the other extreme, $h < -l_f$, $\alpha = \pi_{c1} \cdot (0,0,0,0) = 0$ and $\gamma = \pi_{i2} \cdot (1,1,1,1) = 1$, so $q = 1$: the agent is always in the impulsive mode. Gradually reducing $h$ from $h > \min\{r_f, l_f + \kappa\}$ to $h < -l_f$ increases $\gamma$ and reduces $\alpha$: $h < r_f$ implies the vector of indicators in definition of $\gamma$ has to have 1 as the first entry, $h < r_m < r_f$, implies the first and the third entry has to be unity, further, $h < -l_m < r_m$ implies the first three entries have to be 1, etc. Similarly, reducing $h$ will gradually turn all the indicators in the definition of $\alpha$ to 0 (from 1).

Hence, we have established the following

**Lemma 2.** *The frequency of being in the impulsive regime weakly decreases with the inhibition parameter $h$ in the amnesiac case, i.e., when $\beta = 0$.*

Similar comparisons can be done by changing the payoffs.

As an example, consider lowering the positive payoff, $r_f (> r_m)$. In this case, if $h > 0$, for $r_f$ low enough, the condition $\kappa - r_f > h > r_f$ will hold. It is easy to check that $\alpha = \gamma = 0$, so the regime is switched every period with probability one.[11]

[11] That there is no stationary distribution in the two-state Markov chain IR, CR does not necessarily imply that there is no such distribution for the original eight state chain, note the latter exists by proposition 1.

*Interpretation.* The individual with low (internal) rewards, $r_f$, hardly enjoys his own successes. As a result, (if his memory of past payoffs is short-lived) he is always in doubt, constantly switching from intuition (impulsive regime) to reason (cognitive regime) and back.

Surprisingly, diminishing losses has a different effect, already noted, in part, in lemma 1. If the inhibition is high enough to start with, $h > 1$, then low enough loss will assure condition $h > \kappa + l_f$ holds, which implies $\alpha = 1$, so $q = 0$ (no matter what value $\gamma \in [0, 1]$ attains). In this case, the individual eventually stays in the cognitive regime. If, in addition, $h > r_f$ (as in lemma 1), then, even if the individual will find himself in the impulsive mode (at the beginning, say), he will switch to the cognitive one in the next period. However, if this last inequality is reversed, he might spend several periods (upon success of the impulsive unit) in the impulsive regime, but given $h > l_f + \kappa$ the number of such periods is finite.

If $h < 0$, then diminishing the loss ($|l_f|$) yields $h < -l_f$, so $\gamma = 1$ and thus $q = 1$, for any $\alpha \in [0, 1]$, so the agent is in the impulsive regime with probability one in the long run.

*Interpretation.* The condition for keeping the amnesiac in the cognitive regime, $h > \min\{r_f, l_f + \kappa\}$, requires the inhibition to be high (not surprisingly), but also can be read as making any reward or loss relatively small, perceiving both as relatively unimportant, possibly along the lines of the suggestions given by Kipling (1910) in his poem "If...". The second case, $h < -l_f$, is about the agent with overly powerful impulsiveness, who is driven solely by the "animal spirits".

## 5. General case, $0 < \beta < 1$, heuristics

First step is a heuristic one, based on the stochastic approximation techniques developed by Kushner and Yin (1997), also used by Cho and Matsui (2006).

5.1. **The associated (mean limit) system and its stationary points.** At this point we want to investigate just the behaviour of the average of the original process described in equation (4). The conjecture (that will be confirmed in the next section) is that the asymptotic distribution of the memory will be centered around the stationary point of the associated system of ordinary differential equations (defined below).

In what follows we impose the following assumption (to ensure existence of a non-trivial solution).

**Assumption 1.** [High cognitive cost[12]] $\pi_{i2} \cdot i + \left(\pi_{c1} - \pi_{c2}\right) \cdot c < 0$.

5.1.1. *The associated (mean limit) system.* Define the following system of the ordinary differential equations that describes the behaviour of the mean update of $C, I$:

$$\begin{aligned} &C'_t = \pi_C \cdot c - C_t, && I'_t = \pi_I \cdot i - I_t, && \text{where} \\ &\pi_C = \pi_{c1}, && \pi_I = (0,0,0,0), && \text{if } I_t < C_t + h \\ &\pi_C = \pi_{c2}, && \pi_I = \pi_{i2}, && \text{if } I_t > C_t + h \\ &\pi_C = (1-q)\pi_{c1} + q\pi_{c2}, && \pi_I = q\pi_{i2}, q \in [0,1], && \text{if } I_t = C_t + h \end{aligned} \tag{8}$$

Note that once the system reaches the line $I_t = C_t + h$, its behavior becomes a convex combination of the one above the line (with weight $q$) and that below the line (with weight $1 - q$).

5.1.2. *Stationary solutions.*

**Proposition 2.** *The stationary solution* $(C^*, I^*)$ *of the following system of the ordinary differential equations* (8) *is*

*(K) in case* $\pi_{i2} \cdot i - \pi_{c2} \cdot c < h < -\pi_{c1} \cdot c$, $q^* = \frac{\pi_{c1} \cdot c + h}{\pi_{i2} \cdot i + (\pi_{c1} - \pi_{c2}) \cdot c}$ *and*

$$C^* = q^* \pi_{c2} \cdot c + (1 - q^*)\pi_{c1} \cdot c, \qquad I^* = q^* \pi_{i2} \cdot i$$

*(L) in case* $h > -\pi_{c1} \cdot c$,

$$C^* = \pi_{c1} \cdot c, \qquad I^* = 0$$

*(M) in case* $h < \pi_{i2} \cdot i - \pi_{c2} \cdot c$,

$$C^* = \pi_{c2} \cdot c, \qquad I^* = \pi_{i2} \cdot i$$

*Proof.* Stationarity implies $C_t = C^* = \pi_C \cdot c$ and $I_t = I^* = \pi_I \cdot i$ for all $t$. The equality $I^* = h + C^*$, yields then the formulas for $C^*, I^*$ and $q$ in case (K), and $q^* \in [0,1]$ is assured by the assumption on the range of $h$ and assumption 1.

Similarly, in case (L), the formulas for $C^*, I^*$ satisfy $C^* + h > 0 = I^*$, since in this case $h > -C^*$, and in case (M) $C^* + h < I^*$. □

[12]Interpretation and implications of the assumption are discussed later, in section 7.2.1.

5.1.3. *Stability of the stationary solution.* A direct way to analyze stability of the stationary solution established in proposition 2 is to study the basins of attraction of the solution in each case, and for that one can just solve the system of the ODE using the following lemma.

**Lemma 3.** *Fix a locally integrable real-valued bounded function $A$ defined on $\mathbb{R}$ and $b > 0$. The differential equation $f'_t = A_t - bf_t$ has the following solution*

$$f_t = e^{-b(t-t_0)} f_{t_0} + \int_{t_0}^{t} e^{-b(t-s)} A_s ds \tag{9}$$

*Proof.* By direct computation. □

**Proposition 3.** *The stationary solutions $C^*, I^*$ are globally asymptotically stable.*

*Proof.* By lemma 3, the solution in each of the three cases should be of the form:

$$C_t = e^{-(t-t_0)} C_{t_0} + \pi_C \cdot c \int_{t_0}^{t} e^{-(t-s)} ds \tag{10}$$

$$I_t = e^{-(t-t_0)} I_{t_0} + \pi_I \cdot i \int_{t_0}^{t} e^{-(t-s)} ds \tag{11}$$

Consider the case $h > -\pi_{c1} \cdot c$, the requirement which defines case (L) of proposition 2. If $C_{t_0} + h > I_{t_0}$, then (10)-(11) imply that there is $\hat{t} \leq \infty$ such that starting from $t_0$ up to $\hat{t}$

$$C_t = e^{-(t-t_0)} C_{t_0} + \pi_{c1} \cdot c(1 - e^{-(t-t_0)}) \tag{12}$$

$$I_t = e^{-(t-t_0)} I_{t_0} \tag{13}$$

Observe that the derivative of the difference, $C_t - I_t$, equals

$$e^{-(t-t_0)}(\pi_{c1} \cdot c - (C_{t_0} - I_{t_0})) \tag{14}$$

which is non-negative, whenever $\pi_{c1} \cdot c \geq C_{t_0} - I_{t_0}$. This implies that if $\pi_{c1} \cdot c > C_{t_0} - I_{t_0} > -h$, then $C_t + h > I_t$ for all $t \geq t_0$, hence (12)-(13) applies for all $t \geq t_0$ (so $\hat{t} = \infty$), and the asymptotic solution is

$$\lim_{t\to\infty} e^{-(t-t_0)} I_{t_0} = 0 \tag{15}$$

$$\lim_{t\to\infty} e^{-(t-t_0)} C_{t_0} + \pi_{c1} \cdot c(1 - e^{-(t-t_0)}) = \pi_{c1} \cdot c$$

In other words, the basin of attraction of the above solution includes all pairs of starting values $C_{t_0}, I_{t_0}$ that satisfy $\pi_{c1} \cdot c > C_{t_0} - I_{t_0} > -h$. The interior of this set is non-empty as $h > -\pi_{c1} \cdot c$ in this case.

Consider now the starting value for which the derivative of the difference, (14) is negative, i.e., $C_{t_0} - I_{t_0} > \pi_{c1} \cdot c$. In this case, the difference

$C_t - I_t$ will fall with time, $t$, until it reaches (at some finite time $\tilde{t}$) the value $\pi_{c1} \cdot c$, and from thereon the difference will stay constant, since its derivative (14) is zero, while the solution will converge according to (15) to the stationary one in this case (described in (L) in proposition 2).

Finally, for this case, if $I_{t_0} - C_{t_0} > h$, then (10)-(11) imply that there is $\hat{t} \leq \infty$ such that starting from $t_0$ up to $\hat{t}$

$$C_t = e^{-(t-t_0)} C_{t_0} + \pi_{c2} \cdot c(1 - e^{-(t-t_0)}) \tag{16}$$

$$I_t = e^{-(t-t_0)} I_{t_0} + \pi_{i2} \cdot i(1 - e^{-(t-t_0)}) \tag{17}$$

Since the derivative of the difference $I_t - C_t$ for $0 < t < \hat{t}$ equals

$$e^{-(t-t_0)}(\pi_{i2} \cdot i - \pi_{c2} \cdot c - (I_{t_0} - C_{t_0})) \tag{18}$$

the difference should be decreasing, i.e., $\pi_{i2} \cdot i - \pi_{c2} \cdot c < I_{t_0} - C_{t_0}$, because $I_{t_0} - C_{t_0} > h > -\pi_{c1} \cdot c$, and by assumption 1, $\pi_{i2} \cdot i - \pi_{c2} \cdot c < -\pi_{c1} \cdot c$. Hence from a point $I_{t_0} - C_{t_0} > h$ the difference will decrease, and there is a finite $\hat{t}$ such that $I_{\hat{t}} - C_{\hat{t}} = h$. At that point the left derivative, given by (18) is strictly negative, and the right derivative, which is a convex combination of (18) and the negative of (14), is strictly negative as well. It follows that $I_t - C_t$ will fall below $h$ after $\hat{t}$ and hence follow the path analyzed above (case $C_{t_0} + h > I_{t_0}$).

To sum up, if $h > -\pi_{c1} \cdot c$, then the solution from proposition 2, case (L) is globally asymptotically stable.

Assume now $\pi_{i2} \cdot i - \pi_{c2} \cdot c > h$, as in case (M) of proposition 2. If $\pi_{i2} \cdot i - \pi_{c2} \cdot c > I_{t_0} - C_{t_0} > h$, then the difference, $I - C$, will be increasing (since its derivative, (18) is positive) and hence be above $h$ for all $t > t_0$. If the difference starts above $\pi_{i2} \cdot i - \pi_{c2} \cdot c$, then it initially falls down to this level and then remains non-decreasing, hence, too being above $h$ for all $t > t_0$. If the difference starts below $h$, it follows (12)-(13) and increases up to $h$ (its derivative, by (14) is $C_{t_0} - I_{t_0} - \pi_{c1} \cdot c$, which is positive, since $C_{t_0} - I_{t_0} > -h > \pi_{c2} \cdot c - \pi_{i2} \cdot i > \pi_{c1} \cdot c$, where the last inequality follows from assumption 1). Again, at $h$, the sign of the derivative is unchanged, hence the path crosses the line and follows (12)-(13) as described above (case $I_{t_0} - C_{t_0} > h$). Thus, independently of the initial point, the solution converges to

$$\lim_{t \to \infty} e^{-(t-t_0)} C_{t_0} + \pi_{c2} \cdot c(1 - e^{-(t-t_0)}) = \pi_{c2} \cdot c \tag{19}$$

$$\lim_{t \to \infty} e^{-(t-t_0)} I_{t_0} + \pi_{i2} \cdot i(1 - e^{-(t-t_0)}) = \pi_{i2} \cdot i \tag{20}$$

are all pairs $C_{t_0}, I_{t_0}$ satisfying $I_{t_0} - C_{t_0} > h$, provided $\pi_{i2} \cdot i - \pi_{c2} \cdot c > h$. Therefore the stationary solution described in case (M) of proposition 2 is globally asymptotically stable.

Finally consider now the parameters as in case (K) of proposition 2, i.e., assume $\pi_{i2} \cdot i - \pi_{c2} \cdot c < h < -\pi_{c1} \cdot c$. If the starting point is such that $C_{t_0} + h > I_{t_0}$, then the derivative (14) of the difference, $C_t - I_t$, is negative, therefore, at some finite point $\hat{t}$ in time $C_{\hat{t}} - I_{\hat{t}}$ will equal $-h$. Similarly, if $I_{t_0} - C_{t_0} > h$ by (18), this difference will, at some finite time $\hat{t}$ reach $h$, so that $I_{\hat{t}} - C_{\hat{t}} = h$. Therefore, it is sufficient to consider the starting point $t_0 = \hat{t}$, i.e., $C_{t_0} + h = I_{t_0}$. Then for some $q$

$$\lim_{t \searrow t_0} I'_t - C'_t = -I_{t_0} + q\pi_{i2} \cdot i + C_{t_0} - q\pi_{c2} \cdot c - (1-q)\pi_{c1} \cdot c \tag{21}$$

$$= q\pi_{i2} \cdot i - q\pi_{c2} \cdot c - (1-q)\pi_{c1} \cdot c - h \tag{22}$$

If $q = q^*$ from proposition 2, then the derivative (21) is zero and the solution converges to the stationary point $(C^*, I^*)$ specified in proposition 2 for this case (K).

If $q \neq q^*$, then, we claim, there is a subsequence $z_n = (C_{t_n}, I_{t_n})$, on the line, $C_{t_n} + h = I_{t_n}$, that converges to the same point. Here is why. Consider a point on the line, denote it by $z_0 = (C_{t_0}, I_{t_0}) \neq (C^*, I^*)$ (cf. fig. 5.1.3). As follows from the definition of the mean ODE, eq. (8), for any $q \in [0, 1]$, the right derivative at $z_0$ points to a direction which is a convex combination of the two systems: first, for the half-plane above the line, (16)-(17), and second, below the line, (12)-(13). As it follows from these equations, the first direction is towards point $(\pi_{c2} \cdot c, \pi_{i2} \cdot i)$, denoted by $M$ on the graph, in fig. 5.1.3, and the second direction is towards point $(\pi_{c1} \cdot c, 0)$, $L$ on the graph. Hence the path moves into the simplex formed by $z_0, L, M$. Since $q \neq q^*$, the path will not follow the line towards $K$, but, rather, move towards a different point on the segment containing $L$ and $M$. Assume it moves towards a point above the line $I = C - h$ (the argument for the complementary case is similar), and so passes through a point $w$ (illustrated in fig. 5.1.3 as well). But at any point above the line the process is governed by the system (16)-(17) and so moves towards point $M$, till it reaches the line $I = C - h$. The point, $z_1$, where it hits the line therefore is on the intersection of the segment starting at $w$ and ending at $M$ (this is the direction of the process (16)-(17)) and the line, precisely, the segment $z_0K$. Since $w$ is in the simplex formed by $z_0, L, M$, so is the segment connecting $w$ with the simplex corner, $M$. $K$, the candidate stationary point, too, is in the simplex being a convex combination of $L$ and $M$ by case (K) of proposition 2, so the segment $z_0K$ belongs to the simplex. Hence, the point of intersection $z_1$ between the segments $z_0K$ and $wM$ belongs to the simplex as well. This point, $z_1$, is a convex combination of $z_0$ and $K$ and does not coincide with $z_0$, since $z_0 \neq K$, therefore $z_1$ is closer to $K$ than $z_0$.

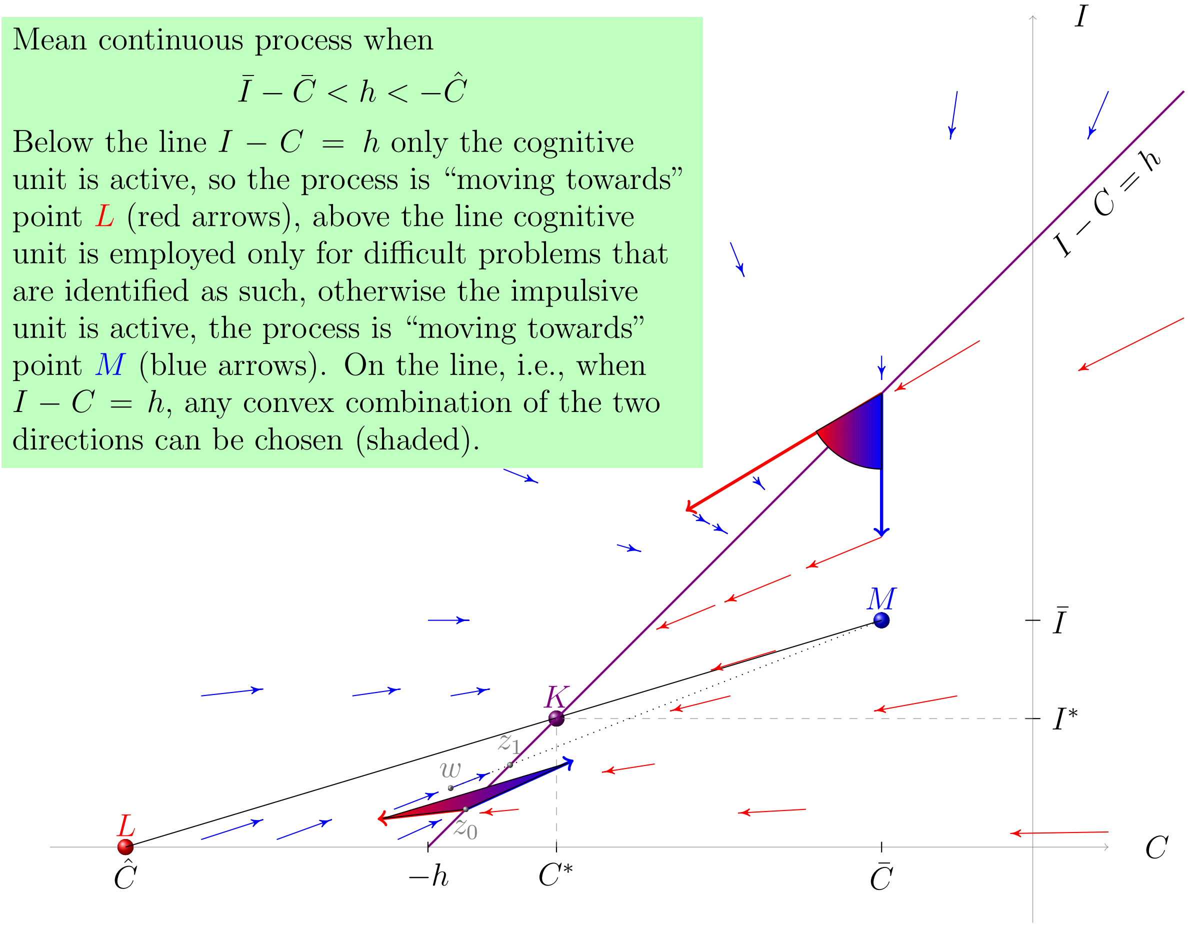


FIGURE 5.1. Stationary point of the mean continuous process is $(C^*, I^*)$.

Let now $(C_{t_1}, I_{t_1}) = z_1$ and find the next point where the process hits the line $I = C - h$ by repeating the same argument.

It follows that the subsequence so constructed converges to $K$.

If the process starts at $K$, then it remains there provided $q = q^*$. For a different $q$, its direction is either towards $M$ or towards $L$, i.e., it stays on the $LM$ segment. But from any point on the segment it is either above the line, hence follows (16) – (17) pointing to $M$, or below the line, hence follows (12) – (13) pointing to $L$, thus returning it to $K$ in both cases. □

## 6. Basic Convergence Result

Let $Z_t$ be the random payoff (column) vector in period $t$, which depends upon the regime where $(C_t, I_t)$ belongs to. If $I_t < C_t + h$

(cognitive regime), then $Z_t = U$ which is a random variable, whose expected value is

$$\bar{U} = \begin{bmatrix} \pi_{c1} \cdot c \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{C} \\ 0 \end{bmatrix}$$

If $I_t > C_t + h$ (impulsive regime), then $Z_t = V$, whose expected value is

$$\bar{V} = \begin{bmatrix} \pi_{c2} \cdot c \\ \pi_{i2} \cdot i \end{bmatrix} = \begin{bmatrix} \bar{C} \\ \bar{I} \end{bmatrix}$$

Finally, if $I_t = C_t + h$, (both regimes are equally good), $Z_t = qU + (1-q)V$ for some $q \in [0,1]$ (in this case we leave $q$ undefined). By construction, the distribution of $Z_t$ is discontinuous at $I_t = C_t + h$. We want to approximate the asymptotic behaviour of the system by the behaviour of a smooth system, i.e., by adding a little "noise" in the decision of the control unit. Of course, to make such an approximation meaningful, we want to rely as little as possible on a particular form of noise that we add, so we impose minimal assumptions for a variation of a standard convergence argument to go through. So, first, we introduce the payoff with noise.

For any $\sigma > 0$, conditional on $y_t \overset{\text{def}}{=} C_t - I_t + h$,

$$Z_t^\sigma = \begin{cases} U & \text{with probability } \ \eta^\sigma(y_t) \\ V & \text{with probability } \ 1 - \eta^\sigma(y_t). \end{cases} \tag{23}$$

To make this definition work, we have to assume that $\eta^\sigma$ is defined on $\mathbb{R}$ and returns values in the unit interval. Next requirement for approximation to work is convergence of $\eta^\sigma$ to a step function (or the indicator, $\mathbb{1}_{y_t>0}$) as $\sigma \to 0$. Requiring convergence of $\eta^\sigma$ in $L_1$ assures that $Z_t^\sigma$ converges to $Z_t$ in distribution (as $\sigma \to 0$). In addition we will also require $\eta^\sigma$ to be continuous and monotone (in its argument $y_t$) for any $\sigma > 0$.[13]

Next, let, as before, $Y_t$ denote the two dimensional real vector $(C_t, Y_t)$. Then the original system can be written as

$$Y_{t+1} = \beta Y_t + (1-\beta) Z_t^\sigma + (1-\beta)\zeta_t^\sigma \tag{24}$$

where $\zeta_t^\sigma \overset{\text{def}}{=} (Z_t - Z_t^\sigma)$ is the "error of approximation". So, without the last term, the system describes evolution of memory with some noise,

[13] Take, for example,

$$\eta^\sigma(y) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{y} e^{\frac{-x^2}{2\sigma^2}} dx.$$

i.e., when the control unit makes "mistakes" with probability that decreases with the distance from the threshold, $\{(C_t, I_t)\colon C_t + h = I_t\}$.

Our objective now is to describe the asymptotic distribution of $Y_t$ defined by the original system (24). To describe distribution of $Y_t$ at any given time $t$, let us define the underlying probability measure.

*Notation.* Let $(\Omega, \mathcal{F}, \mu)$ be a probability space such that $Y_t \in \mathbb{R}^2$ for non-negative $t$ is defined on it. Let $\mathcal{F}_t^\sigma \subset \mathcal{F}$ be the $\sigma$-field generated by $Y_t \stackrel{\text{def}}{=} (Y_0, Z_0^\sigma, \zeta_0^\sigma, Z_1^\sigma, \zeta_1^\sigma \ldots, Z_{t-1}^\sigma, \zeta_{t-1}^\sigma)$, so that $\{\mathcal{F}_t^\sigma, t \in \mathbb{Z}_+\}$ is an increasing family of $\sigma$-fields. Let $\mu_t$ be a probability measure defined on $\mathcal{F}_t^\sigma$ for any $t \geq 0$. Let $\mathsf{E}_t$ be the expectation with respect to $\mu_t$ (or conditional on $Y_t$). Let realizations of all random variables be zero for $t < 0$.

Note that distribution of each of the random variables $Z_t^\sigma$ depends on $y_t = (1, -1)Y_t + h$, hence it is important to understand the asymptotic behaviour of $y_t$. Let us start with its expected value,

$$\mathsf{E}_t(y_{t+1}) = \eta^\sigma(y_t)\hat{C} + (1 - \eta^\sigma(y_t))(\bar{C} - \bar{I}) + h$$

**Lemma 4.** *Define function* $F\colon y \mapsto \eta^\sigma(y)\hat{C} + (1 - \eta^\sigma(y))(\bar{C} - \bar{I}) + h$ *on reals and let* $y^\sigma \in \mathbb{R}$ *be a fixed point of* $F$*, i.e., it solves*

$$F(y) = y. \tag{25}$$

$y^\sigma$ *exists and is unique.*

*Proof.* Function $F$ maps a real number into a convex combination of $\hat{C}$ and $\bar{C} - \bar{I}$ translated by $h$. By assumption 1, $\hat{C} < \bar{C} - \bar{I}$. Recall, $\eta^\sigma$ is a continuous and increasing function, which implies that $F$ is continuous and decreasing, and in addition, it returns values in a compact interval. Hence the difference $F(y) - y$ is strictly decreasing on $\mathbb{R}$ and hence has a single zero, $y^\sigma$, which is also the solution of equation (25). □

**Corollary 1.** $y^\sigma = C^\sigma - I^\sigma + h$*, where*

$$C^\sigma = \eta^\sigma(y^\sigma)\hat{C} + (1 - \eta^\sigma(y^\sigma))\bar{C} \tag{26}$$

$$I^\sigma = (1 - \eta^\sigma(y^\sigma))\bar{I} \tag{27}$$

*and so* $z^\sigma$ *is well defined:*

$$z^\sigma \stackrel{\text{def}}{=} \mathsf{E}_t(Z_t^\sigma | C^\sigma, I^\sigma) = \begin{bmatrix} C^\sigma \\ I^\sigma \end{bmatrix}$$

**Lemma 5.** $y_{t+1} \to y^\sigma$ *a.s. as* $t \to \infty$ *for any* $\sigma > 0$.

*Proof.* By definition of $y_t$,

$$\mathsf{E}_t y_{t+1} = \mathsf{E}_t\big((1, -1)[\beta Y_t + (1 - \beta)Z_t] + h\big) \tag{28}$$

$$= \beta\big((1, -1)Y_t + h\big) + (1 - \beta)\big(\mathsf{E}_t((1, -1)Z_t) + h\big) \tag{29}$$

Note that $(1,-1)Y_t + h = y_t$ and $\mathsf{E}_t((1,-1)Z_t) + h = F(y_t)$ with $F$ as defined in lemma 4. It follows that

$$\mathsf{E}_t y_{t+1} = \beta y_t + (1-\beta)F(y_t) \tag{30}$$

Recall, $F$ is decreasing and has a unique fixed point $y^\sigma$. It follows that $y_t > y^\sigma \iff F(y_t) < F(y^\sigma) = y^\sigma$, so whenever $y_t > y^\sigma$, it is also true that $y_t > F(y_t)$. Therefore if $y_t > y^\sigma$ then $\mathsf{E}_t y_{t+1} < y_t$ and otherwise $\mathsf{E}_t y_{t+1} \geq y_t$. It follows that the stochastic sequences $\max\{y_t - y^\sigma, 0\}$ and $\max\{y^\sigma - y_t, 0\}$ are both positive supermartingales and hence both converge almost sure (and in distribution) to a limit. Both limits have to be zero, otherwise taking limits in (30) will lead to a contradiction. □

Now we can show that $Y_t$ can be approximated by an autoregressive process (AR1).

**Proposition 4.** *$Y_t$ converges in distribution as $t \to \infty$ to a process with a finite mean and a finite variance:*

$$Y_{t+1} = (1-\beta)\Sigma_{j=0}^{\infty}\beta^j \tilde{Z}^\sigma_{t-j} + T^\sigma$$

*where $\tilde{Z}^\sigma$ is distributed as $Z^\sigma_t$ when $C_t = C^\sigma$ and $I_t = I^\sigma$; and where $T^\sigma$ converges in distribution to zero as $\sigma \to 0$.*

*Proof.* Recall that by construction $Z^\sigma_t$ has finite support for any $t$ and is a convex combination of random variables $U$ and $V$ with the corresponding weights $\eta^\sigma(y_t)$ and $(1-\eta^\sigma(y_t))$. By lemma 5 and continuity of $\eta$, distribution of $Z^\sigma_t$ converges (a.s.), as $\eta^\sigma(y_t) \to \eta^\sigma(y^\sigma)$. Denote the limiting random variable by $\tilde{Z}^\sigma$, it follows that its expectation at any time is $z^\sigma$, denote its variance by $\mathbf{v}^\sigma$, which is, obviously, finite too.

The noise term, $\zeta^\sigma_t$ is also a random variable with a finite number of possible realizations and is bounded by the highest (in absolute value) realization of $U$, $V$. Hence $T^\sigma = (1-\beta)\Sigma_{j=0}^{\infty}\beta^j\zeta^\sigma_{t-j}$ is bounded and by construction converges to zero in distribution when $\sigma \to 0$. □

*Remark* 1. It immediately follows that the expectation of $Y_t$ is finite, as the range of possible values is bounded. Given the realizations of $U$, $V$ and hence $\tilde{Z}^\sigma$ and its distribution, which is determined by $\eta^\sigma(y^\sigma)$, one can compute the approximate (up to the error term $T^\sigma$) asymptotic distribution of $Y_t$ directly from the statement of the proposition.

As mentioned above (by construction of $\eta^\sigma$), when $\sigma \to 0$, the error term converges to zero (in distribution). Now it is left to find out what happens to the distribution of $\tilde{Z}^\sigma$ as $\sigma \to 0$, or at least, to its mean.

**Lemma 6.** *$z^\sigma = (C^\sigma, I^\sigma) \to (C^*, I^*)$ as $\sigma \to 0$.*

*Proof.* Consider $(C^*, I^*)$ from proposition 2. Let $y^* = C^* - I^* + h$.

As a first step, we show that $y^\sigma = C^\sigma - I^\sigma + h$ converges to $y^*$ as $\sigma \to 0$. For any $\sigma > 0$, by stationarity, $y^\sigma$ satisfies

$$\eta^\sigma(y^\sigma)\hat{C} + (1 - \eta^\sigma(y^\sigma))(\bar{C} - \bar{I}) + h = y^\sigma \tag{31}$$

Define now $F$ as a function of two variables:

$$F\colon (y, \sigma) \mapsto \eta^\sigma(y)\hat{C} + (1 - \eta^\sigma(y))(\bar{C} - \bar{I}) + h$$

It is decreasing in $y$ and for $y > 0$ it is increasing in $\sigma$, while for $y < 0$ it decreases in $\sigma$.

Pick some $\sigma_0 > 0$. Then, unless $\frac{1}{2}(\hat{C} + \bar{C} - \bar{I}) + h = 0$, two scenarios are possible. Either $y^{\sigma_0} > 0$ or $y^{\sigma_0} < 0$.

Assume $y^{\sigma_0} > 0$. For any $\sigma_1 < \sigma_0$, $F(y^{\sigma_0}, \sigma_1) < F(y^{\sigma_0}, \sigma_0) = y^{\sigma_0}$, hence, $F(y^{\sigma_0}, \sigma_1) - y^{\sigma_0} < 0$. The right hand side of the inequality is a decreasing function of $y$, hence its zero, $y^{\sigma_1}$, should be below $y^{\sigma_0}$. Moreover, $y^{\sigma_1} > 0$, as in the opposite case $F(0, \sigma_1)$ should be negative. However, by monotonicity, $F(y^{\sigma_0}, \sigma_0) - y^{\sigma_0} < F(0, \sigma_1)$ because $y^{\sigma_0} > 0$, which leads to contradiction, because the left hand side of the inequality is zero.

Thus $y^{\sigma_0} > y^{\sigma_1} > 0$ for any $0 < \sigma_1 < \sigma_0$. So, in this case solution $y^\sigma$ is decreasing in $\sigma$, and, by the similar argument, if $y^{\sigma_0} < 0$, the solution is increasing in $\sigma$. Moreover, in each case, since the sign of $y^\sigma$ can not change as $\sigma \to 0$, so that $|y^\sigma|$ decreases along the sequence, the sequence converges.

The limiting point depends on the parameters.

Assume $\hat{C} + h > 0$, as in case (L) of proposition 2, so that $y^* = \hat{C} + h > 0$. By assumption 1, $\hat{C} < \bar{C} - \bar{I}$, so $F(y^*, \sigma) > y^*$ for any $\sigma > 0$. As $F$ is decreasing in the first argument, $y^\sigma > y^* > 0$ for any $\sigma > 0$, and by the argument above, $y^\sigma$ is decreasing as $\sigma \to 0$. In addition, since $y^* > 0$, as $\sigma \to 0$, $F(y^*, \sigma) \to \hat{C} + h = y^*$, so $y^\sigma \to y^*$.

Similarly, if $\bar{C} - \bar{I} + h < 0$, as in case (M) of proposition 2, so that $y^* = \bar{C} - \bar{I} + h < 0$, the corresponding sequence $y^\sigma$ increases and in the limit approaches $y^*$.

So, the remaining case is (K), i.e., when $y^* = 0$. If $\frac{1}{2}(\hat{C} + \bar{C} - \bar{I}) + h = 0$, then $y^\sigma = 0$ for all $\sigma > 0$, and so the limit of this constant sequence is zero as well. Assume now $\frac{1}{2}(\hat{C} + \bar{C} - \bar{I}) + h > 0$, which implies $F(0, \sigma) > 0$ for any $\sigma > 0$ by construction of $\eta^\sigma$. Again, by monotonicity of $F$, it implies that $y^\sigma > 0$ for any $\sigma > 0$. By the above argument, the sequence $y^\sigma$ decreases as $\sigma \to 0$ and remains strictly positive. Moreover, for any $y > 0$ there is $\sigma$ low enough such that $F(y, \sigma) < y$: as $\sigma \to 0$, $F(y, \sigma) \to \hat{C} + h < 0$. It follows that in this case $y^\sigma \to 0$. In case $\frac{1}{2}(\hat{C} + \bar{C} - \bar{I}) + h < 0$, the argument is similar.

To conclude the proof, observe that for any $\sigma > 0$, the stationary solution $(C^\sigma, I^\sigma)$ belongs to the interval with endpoints $\bar{U}$ and $\bar{V}$ (or $L$ and $M$ in the graph 5.1.3) as implied by equations (27)-(26), which have to hold in a stationary solution. Moreover, for any $y^\sigma$, the solution is uniquely determined by these equations. It follows that any infinite sequence $(C^\sigma, I^\sigma)$ constructed for $\sigma \searrow 0$, has to have a converging subsequence. In addition, according to the argument above, $y^\sigma - y^*$ does not change sign along any sequence with $\sigma \to 0$. It implies that the sign of $F(y^\sigma, \sigma) - y^*$ does not change either. It decreases (to zero) if positive and increases (to zero) if negative, hence any sequence $F(y^\sigma, \sigma) - y^*$ with $\sigma \to 0$ converges to zero by the monotonic convergence theorem. Hence, the limit $(C^\#, I^\#)$ of any converging subsequence has to satisfy $C^\# - I^\# + h = y^*$. This condition along with the requirement that $(C^\#, I^\#)$ is a convex combination of $\bar{U}$ and $\bar{V}$, (thus, in particular, belongs to the line passing through the two points) uniquely defines the limiting point (if it exists): linear independence of the two equations is assured by assumption 1. It is easy to check that if $\hat{C} + h > 0$, the solution is $(C^\#, I^\#) = (\hat{C}, 0)$, which is $(C^*, I^*)$ in that case; and if $\bar{C} - \bar{I} + h < 0$, the only solution is $(C^\#, I^\#) = (\bar{C}, \bar{I})$, which, too is $(C^*, I^*)$ in this case. In the complementary case ((K) of proposition 2), $y^* = 0$, so the two conditions ($C^\# - I^\# + h = 0$ and $(C^\#, I^\#) = q\bar{U} + (1-q)\bar{V}$ for some $q \in [0,1]$) yield $(C^*, I^*)$ in this case as well. □

*Remark* 2. The order of limits matters: if $\sigma \to 0$ is taken first and $t \to \infty$ thereafter, the result might differ. Indeed, if $\sigma \to 0$ the fixed point of (25), has to converge to zero, but for any $\sigma > 0$, $\eta^\sigma(0)$ depends on the construction of $\eta$, using the example from footnote 13, $\eta^\sigma(0) = \frac{1}{2}$, so the stationary point in the limit has to be both half-way between $\bar{U}$ and $\bar{V}$ and satisfy $y = 0$, which is possible only for a knife-edge choice of parameters: $\frac{1}{2}(\hat{C} + \bar{C} - \bar{I}) + h = 0$.

**Corollary 2.** *If $t \to \infty$, $\sigma \to 0$ and $\beta \to 1$, then the distribution of $Y_t$ approaches normal with mean $(C^*, I^*)$.*

## 7. Empirical implications and discussion

7.1. **Comparative statics: changes in nature and nurture.** In this section we assume that the assumptions of corollary 2 are satisfied, so, in particular, $\beta$ is close to unity, i.e., the memory (or recall) is almost perfect. In this case, we can study how the asymptotic (predicted) average, $(C^*, I^*)$, is affected by the parameters.

**Proposition 5.** *The asymptotic frequency of using cognitive regime increases if any one of the conditions below holds*

*(A) inhibition parameter, $h$, increases;*
*(B) cost of thinking, $\kappa$, decreases;*
*(C) $z > \frac{b}{b+g}$ and $r_f$ increases;*
*(D) $z > \frac{1-b}{2-b-g}$ and $l_f$ falls.*

*Proof.* As follows from proposition 2, $(C^*, I^*)$ belongs to the interval connecting $(\hat{C}, 0)$ and $(\bar{C}, \bar{I})$. The closer it is to the first point, the more often the cognitive regime (and hence the cognitive unit) is used. So, it is sufficient to show that the point $(C^*, I^*)$ moves closer to $(\hat{C}, 0)$ under each of the conditions stated.

By proposition 2, there are three cases to consider.

First, if $h > \hat{h} \stackrel{\text{def}}{=} -\hat{C}$, then $(C^*, I^*) = (\hat{C}, 0)$. So, in this case, one has to assure that the inequality is not violated, as a result of a change, i.e., either $h$ grows or the threshold, $\bar{h}$ falls.

Second, if $h < \bar{h} \stackrel{\text{def}}{=} \pi_{i2}i - \pi_{c2}c = \bar{I} - \bar{C}$, then $(C^*, I^*) = (\bar{C}, \bar{I})$, so in this case $(C^*, I^*)$ is the furthest it can be from the other end of the segment, $(\hat{C}, 0)$. Moving $h$ into a different region (violating the inequality) can move $(C^*, I^*)$ closer to $(\hat{C}, 0)$ according to proposition 2. This can be accomplished by either increasing $h$ or by lowering the threshold $\bar{h}$.

Finally, consider the intermediate case, $\bar{h} < h < \hat{h}$. The relevant range is non-empty by assumption 1. Then $q = \frac{h+\hat{C}}{\bar{I}-\bar{C}+\hat{C}} = \frac{h-\hat{h}}{\bar{h}-\hat{h}}$ from proposition 2, defines the average of the asymptotic distribution of the indicators: $(C^*, I^*) = q(\bar{C}, \bar{I}) + (1-q)(\hat{C}, 0)$.

**Lemma 7.** *Assume $\bar{h} < h < \hat{h}$. Then $\frac{h-\hat{h}}{\bar{h}-\hat{h}}$ is decreasing in $h$ and is increasing in $\bar{h}$ and $\hat{h}$.*

*Proof.* The first claim holds by assumption 1: $\bar{h}-\hat{h} < 0$; the second one is true since $h - \hat{h} < 0$; and the third one holds because $h - \bar{h} > 0$. □

It follows that any factor decreasing $\hat{h}$ and $\bar{h}$ or an increase in $h$ can not move $(C^*, I^*)$ away from $(\hat{C}, 0)$. So, claim (A) follows.

To demonstrate the rest, we need two auxiliary statements.

**Lemma 8.** $\hat{h} \stackrel{\text{def}}{=} -\hat{C}$ *decreases in rewards $r_f$ and $r_m$ and increases in the cost, $\kappa$, and in the losses, $l_f$, $l_m$.*

*Proof.* The assertion directly follows from the definition:

$$-\hat{C} = w\big(a(\kappa - r_m) + (1-a)(\kappa + l_m)\big) + (1-w)\big(g(\kappa - r_f) + (1-g)(l_f + \kappa)\big)$$

□

**Lemma 9.** $\bar{h} \stackrel{\text{def}}{=} \bar{I} - \bar{C}$ *increases with* $r_m, \kappa$ *and falls with* $l_m$. *It increases with* $r_f$ *iff* $(1-z)b > zg$ *and falls with* $l_f$ *iff* $(1-g)z < (1-z)(1-b)$.

*Proof.* The assertions follow directly from the computation below:

$$\bar{I} - \bar{C} = wpr_m - w(1-p)l_m + (1-w)(1-z)br_f - (1-w)(1-z)(1-b)l_f + (1-w)z((1-g)l_f + g(\kappa - r_f))$$

□

So, statement (B) follows directly from the two lemmata above, since a decrease in $\kappa$ decreases both $\hat{h}$ and $\bar{h}$.

(C): Since $\hat{h}$ decreases in $r_f$ by lemma 8, it is sufficient to assure that $\bar{h}$ falls with $r_f$, which is true if $(1-z)b < zg$ by lemma 9, thus the statement.

(D): Similarly, by lemma 8, $\hat{h}$ increases in $l_f$, and by lemma 9, $\bar{h}$ increases in $l_f$ if $(1-g)z > (1-z)(1-b)$. □

Surprisingly, some of the results from the benchmark case carry over to this case, at least in terms of averages.

Observe that if $h > l_f + \kappa$, as is required in the amnesiac case for cognitive regime to prevail, and assuming $l_f > l_m > 0$, the condition $h > \hat{h} \stackrel{\text{def}}{=} -\hat{C}$ is satisfied (because $l_f + \kappa > -\hat{C}$). Hence lowering losses in this case has the same effect (on the average) as in the benchmark.

In contrast to the benchmark, however, increasing rewards do increase the range of $h$ for which cognitive regime prevails, as follows from proposition 5.

*Interpretation.* Case (A). Inhibition, or strength of the executive control, $h$, is given by "nature" and, as one would expect, the model predicts that the higher the inhibition, the more the cognitive regime is used and hence fewer decisions are made impulsively.

Case (B) demonstrates that increasing the cost of thinking decreases the frequency of its use. Importantly, no "conscious" decision is involved and no meta-calculations are performed to assess the cost-benefit analysis of thinking. The result is caused by the reinforcement-learning-based trigger activating the best-performing unit.

Cases (C)-(D). Even if inhibition is not very high, but the agent is alert and identifies difficult problems frequently, then increasing the "carrot" and reducing the "stick" enhances more thoughtful or contemplative decisions, which are made by the cognitive unit, cf. the motivating example at the beginning of the paper.

### 7.2. **Discussion.**

7.2.1. *The high cognitive costs assumption.* Recall that all the results we got are subject to assumption 1: $\bar{I} < \bar{C} - \hat{C}$.

First, to justify the name of the assumption, notice that the left hand side of the inequality is independent of $\kappa$, whereas the difference on the right hand side is increasing in $\kappa$:

$$\begin{aligned}\bar{C} - \hat{C} = (\pi_{c2} - \pi_{c1})c &= wa(\kappa - r_m) + w(1-a)(\kappa + l_m) \\ &+ (1-w)g(1-z)(\kappa - r_f) + (1-w)(1-g)(1-z)(\kappa + l_f)\end{aligned}$$

While there are many reasons to believe that the cost $\kappa$ for thinking about a problem is not neglible (see section 2.1.1 in the introduction), thus, in a way, supporting the assumption, one might still wonder what happens if it is violated.

By inspecting the proof of proposition 2, one can easily see that the same argument holds in the two extreme cases. If (M) is true, but not (L) (the stationary point then is $(\bar{C}, \bar{I})$, that of the impulsive regime); and if (L) is true, but not (M) (the stationary point then is $(\hat{C}, 0)$, that of the cognitive regime). However, if both conditions (L) and (M) hold, then there are two stable points with the region of attraction being the corresponding half-plane (above and below the line $C + h = I$ for $(\bar{C}, \bar{I})$ and $(\hat{C}, 0)$ correspondingly). Most likely, in the latter case the results will have to be formulated for the memory starting in some "locality" of the corresponding stable point, although, apart from this case, we conjecture, the results would remain (qualitatively) the same.

7.2.2. *An uptight dimwit and a laid-back processor.* One might entertain a view that using cognitive regime more, or thinking about problems more frequently should, in some way, improve well-being of the decision-maker. But that would be misleading: first, we did not make any claims about expressing the DM's preferences and second, no assumptions have been made about cognitive unit being more successful in solving problems of any kind.

To illustrate, graph 5.1.3 depicts a story of an uptight dimwitted DM: in intuitive regime he is more successful (on average) in solving problems. Indeed, when only cognitive regime is used, his average reward is $\hat{C} = \pi_{c1}c$, corresponding to the red dot in the graph. However, in the impulsive regime it is $\bar{I} + \bar{C} = \pi_{i2}i + \pi_{c2}c$, the sum of the coordinates of the blue dot. But $\bar{C} + \bar{I} > \bar{C} - \bar{I}$ since in this case $\bar{I} > 0$ and $\bar{C} - \bar{I} > \hat{C}$ by assumption 1. So, one can call the DM "dimwitted". Note that despite the relative success of the impulsive regime, the DM is "stuck" in a stable point $K$ that describes his asymptotic average and this is because his inhibition $h$ is high, which forces him to think too much! Being less uptight (lowering $h$ if possible) and thus "over-thinking" less, can help the DM to gain higher average reward, and if this is a measure of well-being, then it should make him happier too.

It is not hard to construct an example of a DM with high thinking capabilities and poor intuition, call him a relaxed wise-guy. Take, for example a DM with $\bar{I} < 0$ to reflect the poor intuition, this holds if the impulsive unit is rarely successful in solving either type of problem: $p < \frac{l_m}{l_m + r_m}, b < \frac{l_f}{l_f + r_f}$. In addition, assure that his cognitive unit is finding the right solution sufficiently often, $a > \frac{l_m}{l_m + r_m}, g > \frac{l_f}{l_f + r_f}$, thus assuring $\hat{C} > \bar{C}$. Keep the cognitive costs high to assure that assumption 1 is satisfied ($\hat{C} - \bar{C} > -\bar{I}$), so the DM is sufficiently "lazy". Finally, the inhibition parameter $h$ has to be in the intermediate region, $\bar{h} < h < \hat{h}$, so the wise-guy is rather relaxed. Since assumption 1 still holds, our results apply, hence with perfect recall the asymptotic distribution will be centered around the intersection point between the line $C + h = I$ and the segment connecting $(\hat{C}, 0)$ and $(\bar{C}, \bar{I})$. In this case, however, the average payoff in cognitive regime, $\hat{C}$ is higher than that in the impulsive, $\bar{C} + \bar{I}$ (by construction, $\hat{C} > \bar{C} > \bar{C} + \bar{I}$), thus the relaxed wise-guy could have had a higher average reward with a higher inhibition $h$ (by proposition 5), thinking more often and so erring less.

## 8. Conclusions

We have offered a description of a technology of decision making involving a simple mechanism with three units: cognitive (contemplative), impulsive (fast, intuitive) and executive control triggering either of the two units to make the decision.

In a stationary Markovian environment, the model yields predictions of individual behaviour summarised in proposition 5: higher inhibition, lower cost of thinking and, in case of high alertness, higher rewards for solving novel problems as well as lower losses, all enhancing more frequent thinking, provided the memory is almost perfect. This might be a welcome change, at least for the relaxed wise-guy described just above, since "...the sleep of reason produces monsters".[14] However, in some cases, as in the story of the uptight dimwit, an awakened mind might be a worse alternative.

[14] *El sueño de la razon produce monstruos*, as claimed by Francisco de Goya.

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