Equilibria Under Monetary and Fiscal Policy Interactions in a Portfolio Choice Model - Technical Appendix

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Preliminaries

Definition (the index of a fixed point) [Hirsch and Smale (1976)] *Let $\overline{x} \in \mathbb{R}^n$ be a hyperbolic equilibrium; that is, the eigenvalues of $Df(\overline{x})$ have nonzero real parts. In this case, the index $ind(\overline{x})$ of $\overline{x}$ is the number of eigenvalues (counting multiplicities) of $Df(\overline{x})$ having negative real parts.*

The Stable Manifold Theorem [Guckenhaimer and Holmes (1983) Theorem 1.3.2] *Suppose that $\dot{x} = f(x)$ has a hyperbolic fixed point $\overline{x}$. Then there exist local stable and unstable manifolds $W^s_{loc}(\overline{x}), W^u_{loc}(\overline{x})$, of the same dimensions n_s, n_u as those of the eigenspaces E^s, E^u of the linearized system, respectively, and tangent to E^s, E^u at $\overline{x}$. $W^s_{loc}(\overline{x}), W^u_{loc}(\overline{x})$ are as smooth as the function f.*

The Hartman-Grobman Theorem [Guckenhaimer and Holmes (1983) Theorem 1.3.1] *If $Df(\overline{x})$ has no zero or purely imaginary eigenvalues, then there is a homeomorphism h defined on some neighborhood U of $\overline{x}$ in $\mathbb{R}^n$ locally taking orbits of the nonlinear flow Φ_t of $\dot{x} = f(x)$ to those of the linear flow $e^{tDf(\overline{x})}$ of $\dot{y} = Df(\overline{x})y$. The homeomorphism preserves the sense of orbits and can also be chosen to preserve parametrization by time.*

The index of a hyperbolic fixed point is the dimension of the stable manifold. In the context of our model, and given that we have two predetermined variables, equilibrium $\overline{x}$ is determinate if and only if $ind(\overline{x}) = 2$. The implications for our model appear in Tables A.1 and A.2 below, where rr_i denotes the real part of eigenvalue $r_{i \cdot \cdot i=1,..,4}$.

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Table A.1: Index and equilibria in a four-dimensional vector space with two predetermined variables

Sign(rr_1)	Sign(rr_2)	Sign(rr_3)	Sign(rr_4)	det	Trace	Index	Equilibrium
$+$	$+$	$+$	$+$	> 0	> 0	0	no-equilibrium
$-$	$+$	$+$	$+$	< 0	$\gtrless 0$	1	no-equilibrium
$-$	$-$	$+$	$+$	> 0	$\gtrless 0$	2	unique
$-$	$-$	$-$	$+$	< 0	$\gtrless 0$	3	multiple
$-$	$-$	$-$	$-$	> 0	< 0	4	multiple

Table A.2: Index and equilibria in a three-dimensional vector space with two predetermined variables

Sign(rr_1)	Sign(rr_2)	Sign(rr_3)	det	Trace	Index	Equilibrium
$+$	$+$	$+$	> 0	> 0	0	no-equilibrium
$-$	$+$	$+$	< 0	$\gtrless 0$	1	no-equilibrium
$-$	$-$	$+$	> 0	$\gtrless 0$	2	unique
$-$	$-$	$-$	< 0	< 0	3	multiple

A linear approximation to eq. (17)-(20) near a hyperbolic fixed point A linear approximation near the steady state reads

$$\begin{aligned}
\dot{x}_t &= B \times (x_t - x^*) \qquad \text{(A1)} \\
B &\equiv \begin{bmatrix} 0 & -\frac{\sigma\alpha\widetilde{c}^*(\rho+\delta)}{\nu+R^*} f^* & -\frac{\sigma\widetilde{c}^*}{1+\frac{1}{\nu}R^*} f_k^{*2}\left(\beta + \frac{\tau^*}{1-\varphi^*}\right) & -\gamma\frac{\sigma(\rho+\delta)}{1-\tau^*}\frac{\widetilde{c}^*}{\widetilde{a}^*} \\ 0 & \rho+\delta+(\nu+R^*)\frac{\alpha-1}{\alpha} & \frac{\nu}{\alpha}\frac{f_k^{*2}}{f^*}\left(\beta+\frac{\tau^*}{1-\varphi^*}\right) & \gamma\frac{\nu}{\alpha}\frac{f_k^*}{f^*}\frac{1}{\widetilde{a}^*} \\ -1 & 0 & f_k^* - \delta & 0 \\ 0 & \alpha f^*\left[\frac{\alpha-1}{\alpha}\tilde{a}^* - \frac{1}{\nu}\right] & -f_k^*\left[\frac{1}{\nu}R^* + \beta + \tau^*\right] & \rho - \frac{\gamma}{\widetilde{a}^*} \end{bmatrix} \\
x_t &\equiv \begin{bmatrix} c_t \\ \pi_t \\ k_t \\ a_t \end{bmatrix} \quad x^* \equiv \begin{bmatrix} c^* \\ \pi^* \\ k^* \\ a^* \end{bmatrix}.
\end{aligned}$$

(Asterisks denote steady-state levels; $f_k^*, f^*, \tilde{a}^*, \widetilde{c}^*$ are marginal product of capital, GDP, debt-to-GDP, and consumption-to-GDP, respectively; and k_t, a_t are predetermined state variables.)

We obtain analytically that the determinant of B is
$-\left[\widetilde{c}^*\nu\sigma\rho f_k^{*2}\right]\frac{\alpha-1}{\alpha}\left[\beta + \frac{\tau^*}{1-\varphi^*} + \frac{\gamma}{\rho\tilde{a}^*}\left(\frac{1}{\nu}R^* - \frac{\tau^*\varphi^*}{1-\varphi^*}\right)\right]$,
and that the trace of B is

$$2\rho + (\nu + R^*)\frac{\alpha-1}{\alpha} + (\rho+\delta)\frac{1+\frac{1}{\nu}R^*}{1-\tau^*} - \frac{\gamma}{\tilde{a}^*}.$$

Proof of Proposition 4 Assume that $\phi_\alpha \left[\phi_\beta + \phi_\gamma \left(\mathcal{RSF}^* - 1\right) \frac{\tau^*}{1-\varphi^*}\right] \neq 0$. Then the product of eigenvalues is nonzero, which indicates that there is no zero eigenvalue. Assume now that $\phi_\alpha \left[\phi_\beta + \phi_\gamma \left(\mathcal{RSF}^* - 1\right) \frac{\tau^*}{1-\varphi^*}\right] = 0$. Then either $\phi_\alpha = 0$ or $\left[\phi_\beta + \phi_\gamma \left(\mathcal{RSF}^* - 1\right) \frac{\tau^*}{1-\varphi^*}\right] = 0$. In what follows I show that either policy induces a zero eigenvalue, i.e., that there is a bifurcation at $\phi_\alpha = 0$ and given ϕ_β there is a bifurcation at $\phi_\gamma = \frac{\phi_\beta}{-(\mathcal{RSF}^*-1)\frac{\tau^*}{1-\varphi^*}}$.

Assume $\phi_\alpha = 0$ and $\left[\phi_\beta + \phi_\gamma \left(\mathcal{RSF}^* - 1\right) \frac{\tau^*}{1-\varphi^*}\right] \neq 0$.

Substituting $\phi_\alpha = 0$ into equation (A1) we obtain that

$$B_{[\phi_\alpha = 0]} \equiv \begin{bmatrix} 0 & -\frac{\sigma\alpha\widetilde{c}^*(\rho+\delta)}{\nu+R^*} f^* & -\frac{\sigma\widetilde{c}^*}{1+\frac{1}{\nu}R^*} f_k^{*2} \left(\beta + \frac{\tau^*}{1-\varphi^*}\right) & -\gamma \frac{\sigma(\rho+\delta)}{1-\tau^*} \frac{\widetilde{c}^*}{\widetilde{a}^*} \\ 0 & \rho+\delta & \frac{\nu}{\alpha} \frac{f_k^{*2}}{f^*} \left(\beta + \frac{\tau^*}{1-\varphi^*}\right) & \gamma \frac{\nu}{\alpha} \frac{f_k^*}{f^*} \frac{1}{\widetilde{a}^*} \\ B_{3,1} & 0 & B_{3,3} & 0 \\ 0 & B_{4,2} & B_{4,3} & B_{4,4} \end{bmatrix}$$

where $B_{i,j}$ $i, j = 1, ..4$ are components of B specified in eq. (A1), respectively. Where $\alpha = 1$, and hence $\phi_\alpha = 0$, the first row is a multiplication of the second row by $-\frac{\sigma\alpha\widetilde{c}^*}{\nu+R^*} f^*$. Consequently $B_{[\phi_\alpha = 0]}$ is singular.

Assume $\phi_\alpha \neq 0$ and $\left[\phi_\beta + \phi_\gamma \left(\mathcal{RSF}^* - 1\right) \frac{\tau^*}{1-\varphi^*}\right] = 0$.

Substituting $\phi_{\widetilde{\gamma}} = \frac{\phi_\beta}{-(\mathcal{RSF}^*-1)\frac{\tau^*}{1-\varphi^*}}$ into equation (A1) we obtain that

$$B_{\left[\phi_{\widetilde{\gamma}}\right]} \equiv \begin{bmatrix} 0 & B_{1,2} & B_{1,3} & \psi B_{1,3} \\ 0 & B_{2,2} & B_{2,3} & \psi B_{2,3} \\ B_{3,1} & 0 & B_{3,3} & 0 \\ 0 & B_{4,2} & B_{4,3} & \psi B_{4,3} \end{bmatrix}$$

where $\psi \equiv \frac{\rho}{-f_k^*(\mathcal{RSF}^*-1)\frac{\tau^*}{1-\varphi^*}}$ is a constant. It is straightforward to notice that the determinant of $B_{\left[\phi_{\widetilde{\gamma}}\right]}$ equals zero. Thus, a monetary-fiscal regime such that $\phi_\alpha \left[\phi_\beta + \phi_\gamma \left(\mathcal{RSF}^* - 1\right) \frac{\tau^*}{1-\varphi^*}\right] = 0$ brings about a non-hyperbolic equilibrium. This concludes the first part of the proof.[1]

The second part is straightforward. From Table A.1 it follows that a necessary condition for equilibrium determinacy is $\det(B) > 0$. The proof of the proposition is concluded by requiring that the right-hand side of equation (32) is positive.

[1] Assuming $\phi_\alpha = 0$ and $\left[\phi_\beta + \phi_\gamma \left(\mathcal{RSF}^* - 1\right) \frac{\tau^*}{1-\varphi^*}\right] = 0$ simultaneously, brings about a codimension two bifurcation.

Preliminaries to section 4 Let C be a closed convex subset of a Banach space E. A mapping T on C is called a non-expanding mapping if $\|T(x)-T(y)\| \leq \|x-y\|$ for all $x, y \in C$. We denote by $\{\overline{x}\}$ the set of fixed points of T. If C is compact, then Schauder's fixed-point theorem yields that $\{\overline{x}\}$ is non-empty. Obtaining members of $\{\overline{x}\}$ is then feasible via a Krasnoselski sequence. Krasnoselski's Theorem is the following result:

Krasnoselski's Theorem If C is a convex, bounded subset of a uniformly convex Banach space and if T is a mapping of C into a compact subset of C such that $\|T(x)-T(y)\| \leq \|x-y\|$, then the sequence obtained by choosing x_1 in C and defining $x_{n+1} = \frac{1}{2}\left[x_n + T(x_n)\right]$ converges to some z in C and $T(z) = z$.

Mann's (1953) theorem states that if T is a continuous function, not necessarily non-expanding, that takes a closed interval of the real line, $[a, b]$, into itself and has a unique fixed point, z, in $[a, b]$, then there is a sequence that converges to z for all choices of x_1 in $[a, b]$. Mann's Theorem is the following:

Mann's Theorem If $T(x)$ is a continuous function carrying the interval $[a, b]$ into itself and having a unique fixed point, z, on $[a, b]$, then (x_1, Λ, T) converges to z for all choices of x_1 on $[a, b]$, where (x_1, Λ, T) denotes the process of starting with an arbitrary point $x_1$in $[a, b]$ and applying the formulas $x_{n+1} = T(v_n)$ and $v_n = \frac{1}{n}\sum_{i=1}^{n} x_i$.

Bailey (1974) offered a proof for the case in which C is a closed interval of the real line and T is a non-expanding mapping. Bailey's Theorem is the following:

Bailey's Theorem If T takes $[a, b]$ into itself and $\|T(x)-T(y)\| \leq \|x-y\|$, then the sequence obtained by choosing any x_1 in $[a, b]$ and defining $x_{n+1} = \frac{1}{2}\left[x_n + T(x_n)\right]$ converges to some z in $[a, b]$ and $T(z) = z$.

Proof of Lemma 1 In what follows we assume that the government has access to lump-sum taxation and that the income-tax rate is set to zero at all times. Then, the household's lifetime maximization problem becomes

$$V\left[b_t, m_t, k_t\right] = Max_{\{c_s, I_s, x_s\}_{s=t}^{\infty}} \int_t^{\infty} e^{-\rho s} u(c_s) ds$$

$$\begin{aligned} & s.t. \\ \dot{b}_s &= (R_s - \pi_s) b_s - \pi_s m_s + f(k_s) + T_s - I_s - c_s - x_s - \tau_s^L \\ \dot{m}_s &= x_s \\ \dot{k}_s &= I_s - \delta k_s \\ c_s + I_s &\leq \nu m_s \\ a_s, k_s &\geq 0 \end{aligned}$$

with a borrowing constraint such that $lim_{t\to\infty} a_t e^{-\int_0^t [R_s - \pi_s] ds} \geq 0$ where $a_t \equiv b_t + m_t$ denotes the representative household's non-capital wealth, and τ_s^L is a lump-sum tax.

The optimal program: Each household chooses sequences of $\{c_t, I_t, x_t\}$ so as to maximize its lifetime utility, taking as given the initial stock of capital k_0, the initial stock of financial wealth a_0, and the time path $\left\{\tau_t^L, T_t, R_t, \pi_t\right\}_{t=0}^{\infty}$, which is exogenous from the viewpoint of a household. The necessary conditions for an interior maximum are

$$u'(c_t) = \mu_t \tag{A2.1}$$

$$\mu_t = \lambda_t (1 + \frac{1}{\nu} R_t) \tag{A2.2}$$

$$\zeta_t = \frac{1}{\nu} R_t \lambda_t \tag{A2.3}$$

$$\zeta_t (\nu m_t - c_t - I_t) = 0; \zeta_t \geq 0 \tag{A2.4}$$

where λ_t, μ_t are time-dependent co-state variables interpreted as the marginal valuations of financial wealth and capital, respectively, and ζ_t is a time-dependent Lagrange multiplier associated with the liquidity constraint. Assuming a positive nominal rate of interest implies that the liquidity constraint is binding. Second, and after substituting $m_t = \frac{1}{\nu}(c_t + I_t)$ and $a_t = b_t + m_t$ into the household's budget constraint, the state and co-state variables must evolve according to

$$\frac{\dot{\mu}_t}{\mu_t} = \rho + \delta - \frac{1}{1+\frac{1}{\nu}R_t} f'(k_t) \quad \text{(A3.1)}$$

$$\frac{\dot{\lambda}_t}{\lambda_t} = \rho + \pi_t - R_t \quad \text{(A3.2)}$$

$$\dot{k}_t = I_t - \delta k_t \quad \text{(A3.3)}$$

$$\dot{a}_t = (R_t - \pi_t)a_t + f(k_t) + T_t - \tau_t^L - (c_t + I_t)\left(1 + \frac{1}{\nu}R_t\right). \quad \text{(A3.4)}$$

The household's intertemporal budget constraint is of the form

$$lim_{t\to\infty} e^{-\int\limits_0^t [R_s - \pi_s]ds} a_t = a_{0+} \int\limits_0^{\infty} e^{-\int\limits_0^t [R_s - \pi_s]ds} \left[f(k_t) + T_t - \tau_t^L - (c_t + I_t)\left(1 + \frac{1}{\nu}R_t\right)\right] dt \geq 0 \quad \text{(A4)}$$

and the condition that its intertemporal budget constraint holds with equality yields the usual transversality condition:

$$lim_{t\to\infty} a_t e^{-\int\limits_0^t [R_s - \pi_s]ds} = 0. \quad \text{(A5)}$$

The government and the evolution of government debt: With lump-sum taxation, government liabilities evolve according to:

$$\dot{a}_t = (R_t - \pi_t)\, a_t - R_t m_t + \left(T_t - \tau_t^L\right) \quad \text{(A6)}$$

and we assume that the fiscal rule emphasizes output targeting and debt targeting,

$$\tau^L(k_t, a_t) = \tau^{L*} + \beta\,[f(k_t) - f(k^*)] + \gamma\,[a_t - a^*] \quad \text{(A7.1)}$$

$$\forall t \; : \; T_t = \overline{T} \quad \text{(A7.2)}$$

where $f(k^*), a^*$ are output and debt targets, respectively, and $\overline{T}$ is an exogenous flow of lump-sum transfers.

General Equilibrium: In equilibrium, the goods market clears,

$$f(k_t) = c_t + I_t, \quad \text{(A8)}$$

the money market clears,

$$m_t = \frac{1}{\nu}\left(c_t + I_t\right), \tag{A9}$$

and government liabilities equal household assets. Using the monetary policy rule, imposing market clearing conditions, and assuming that the elasticity of intertemporal substitution in consumption is constant, the characterization of the general equilibrium of the economy becomes

$$\frac{\dot{c}_t}{c_t} = \sigma\left[\frac{1}{1+\frac{1}{\nu}R\left(\pi_t\right)}f'(k_t) - (\rho+\delta)\right] \tag{A10.1}$$

$$\dot{\pi}_t = \frac{\nu + R\left(\pi_t\right)}{\alpha}\left\{\left[R\left(\pi_t\right) - \pi_t\right] - \left[\frac{1}{1+\frac{1}{\nu}R\left(\pi_t\right)}f'(k_t) - \delta\right]\right\} \tag{A10.2}$$

$$\dot{k}_t = f(k_t) - c_t - \delta k_t \tag{A10.3}$$

$$\dot{a}_t = \left[R\left(\pi_t\right) - \pi_t\right]a_t - \frac{1}{\nu}R\left(\pi_t\right)f(k_t) + \left[\overline{T} - \tau^L(k_t, a_t)\right] \tag{A10.4}$$

and we define a perfect-foresight equilibrium with lump-sum taxation as a set of sequences $\left\{c_t, \pi_t, k_t, a_t, \tau_t^L, T_t, R_t\right\}$ and an initial price level $P_0 > 0$ satisfying (A8)-(A10.4) given $M_0 + B_0 > 0$ and $k_0 > 0$.

Steady-State Equilibrium: It follows from equation (A10.1) that in a steady state

$$f'(k^*) = (\rho+\delta)\left(1 + \frac{1}{\nu}R^*\right) \tag{A11}$$

where R^* is a steady state rate of interest. From equations (A10.2) and (A11), R^* must satisfy

$$R^* = \rho + \pi^* \tag{A12}$$

where π^* is the long-run rate of inflation. Equation (A10.3) implies that the steady state consumption is

$$c^* = f(k^*) - \delta k^*. \tag{A13}$$

Finally, it follows from equation (A10.4) that in a steady-state equilibrium, government liabilities must satisfy $a^* = \frac{1}{\rho}\left[f(k^*)\frac{1}{\nu}R^* + \tau^{L*} - \overline{T}\right]$. Let $\widetilde{a}^* \equiv \frac{a^*}{f(k^*)}$, $\widetilde{s}^* \equiv \frac{\tau^{L*} - \overline{T}}{f(k^*)}$ denote liabilities to GDP and surplus to GDP in the steady state, respectively. Then we obtain that a sustainable debt level must satisfy

$$\widetilde{a}^* = \frac{1}{\rho}\left[\frac{1}{\nu}R^* + \widetilde{s}^*\right]. \tag{A14}$$

Equilibrium Dynamics Solving equation (A10.4) and letting $t \to \infty$ yields the well-known assertion that market equilibrium requires an intertemporal government budget balance:

$$0 = \lim_{t\to\infty} e^{-\int_0^t [R(\pi_s)-\pi_s]ds} a_t = a_0 - \int_0^\infty e^{-\int_0^t [R(\pi_s)-\pi_s]ds} \left\{ \frac{1}{\nu} R(\pi_t) f(k_t) + \tau^L(k_t, a_t) - \overline{T} \right\} dt.$$

From a) solving equation (A10.4), which internalizes the idea that in equilibrium, households' assets equal the government's liabilities, and b) imposing conditions (A4)-(A5), it follows that the households' intertemporal budget constraint holds with equality. Note that substituting the fiscal rule (A7.1) into (A10.4) yields that government liabilities evolve according to:

$$\dot{a}_t = [R(\pi_t) - \pi_t - \gamma] a_t - f(k_t) \left[\frac{1}{\nu} R(\pi_t) + \beta \right] + \left[\overline{T} - \tau^{L*} + \beta f(k^*) + \gamma a^* \right]. \qquad \text{(A15)}$$

Solving equation (A15) for a_t we obtain that:

$$Q_t a_t = a_0 - \int_0^t Q_s \left\{ f(k_s) \left[\frac{1}{\nu} R(\pi_s) + \beta \right] + S^{**} \right\} ds \qquad \text{(A16)}$$

where $Q_t \equiv e^{-\int_0^t [R(\pi_s)-\pi_s-\gamma]ds}$ is a discount factor and $S^{**} \equiv \tau^{L*} - \left[\overline{T} + \beta f(k^*) + \gamma a^* \right]$ sums all the constant terms in eq. (A15). Letting $t \to \infty$ and rearranging we obtain that:

$$\frac{B_0+M_0}{P_0} = \int_0^\infty Q_s \left\{ f(k_s) \left[\frac{1}{\nu} R(\pi_s) + \beta \right] + S^{**} \right\} ds.$$

When the government operates a fiscal rule with $\gamma > \rho$ it ensures that its liabilities will converge back to the target, and for any price level, fiscal solvency is ensured by the fiscal policy. In this case fiscal policy is considered Ricardian, and the level of nominal prices is determined so as to clear the money market. Specifically, where $\gamma > \rho$, P_0 is determined according to equation (A9), i.e., $\frac{M_0}{P_0} = \frac{1}{\nu}(c_0 + I_0)$. Where $\gamma < \rho$ fiscal policy is considered non-Ricardian and P_0 must jump so as to restore equilibrium.

Transitional Dynamics A linear approximation to equations (A10.1)-(A10.4) near the steady state is obtained through the system

$$\dot{x}_t = A \times (x_t - \overline{x}) \qquad \text{(A17)}$$

where[2]

[2] Asterisks denote steady-state levels.

$$x_t \equiv \begin{bmatrix} c_t \\ \pi_t \\ k_t \\ a_t \end{bmatrix} \quad \overline{x} \equiv \begin{bmatrix} c^* \\ \pi^* \\ k^* \\ a^* \end{bmatrix} \quad A \equiv \begin{bmatrix} 0 & -\frac{\sigma\alpha\widetilde{c}^*(\rho+\delta)}{\nu+R^*} f^* & -\frac{\sigma\widetilde{c}^*}{1+\frac{1}{\nu}R^*}\frac{f_k^{*2}}{\xi_k} & 0 \\ 0 & \rho+\delta+(\nu+R^*)\frac{\alpha-1}{\alpha} & \frac{\nu}{\alpha}\frac{f_k^{*2}}{f^*\xi_k} & 0 \\ -1 & 0 & f_k^*-\delta & 0 \\ 0 & f^*\left[(\alpha-1)\tilde{a}^*-\frac{\alpha}{\nu}\right] & -f_k^*\left[\frac{1}{\nu}R^*+\beta\right] & \rho-\gamma \end{bmatrix}$$

where π^* is a policy target proclaimed by the government denoting the long-run level of inflation, c^*, k^*, a^* are steady-state levels obtained by equations (A11)-(A14), $f_k^*, f^*, \tilde{a}^*, \widetilde{c}^*$ are marginal product of capital, GDP, debt to GDP, and consumption to GDP, respectively, and $\xi_k \equiv \frac{[f'(k^*)]^2}{f(k^*)f''(k^*)} < 0$ is the (constant) elasticity of production technology. Let $r_i \quad i = 1,..,4$ denote the eigenvalues of A. Then by calculating the determinant and trace of A we obtain that:

$$r_1 r_2 r_3 r_4 = -\frac{\sigma\widetilde{c}^*(\nu+R^*)^2(\rho+\delta)^2}{\alpha\nu\xi_k}(\alpha-1)(\gamma-\rho) \tag{A18}$$

$$r_1+r_2+r_3+r_4 = -\gamma+2\rho+(\nu+R^*)\frac{\alpha-1}{\alpha}+(\rho+\delta)(1+\frac{1}{\nu}R^*). \tag{A19}$$

Obviously, the steady state $\overline{x}$ is hyperbolic if and only if $(\alpha-1)(\gamma-\rho) \neq 0$. *Furthermore,* a necessary condition for determinacy of equilibrium is
$(\alpha-1)(\gamma-\rho) > 0$.

Stabilizing Monetary-Fiscal Interactions with Lump-Sum Taxes and Finance Constraints: Examining eq. (A17), the dynamics of (c, π, k) are independent of government liabilities. This feature has two implications: a) one eigenvalue of the (c, π, k, a) system is $\rho-\gamma$; b) the remaining three eigenvalues are determined by $\widehat{A}$, the upper left 3×3 submatrix of A, so that the dynamics of (c, π, k) are completely determined by $\widehat{A}$. It is straightforward to show that the three remaining eigenvalues satisfy:

$$r_1 r_2 r_3 = \frac{\sigma\widetilde{c}^*(\nu+R^*)^2(\rho+\delta)^2}{\alpha\nu\xi_k}(\alpha-1) \tag{A20}$$

$$r_1+r_2+r_3 = \rho+(\nu+R^*)\frac{\alpha-1}{\alpha}+(\rho+\delta)(1+\frac{1}{\nu}R^*). \tag{A21}$$

And since the fourth eigenvalue equals $\rho-\gamma$ we are able to obtain Leeper's (1991) result for this economy:

Proposition *Two monetary-fiscal regimes induce a determinate equilibrium:*

I) Active-Monetary Passive-Fiscal where $\alpha > 1$ and $\gamma > \rho$. In this regime nominal prices are pinned down so as to clear the money market.

II) *Passive-Monetary Active-Fiscal where $\alpha < \frac{1}{1+\frac{\rho}{\nu+R^*}+\frac{\rho+\delta}{\nu}}$ and $\gamma < \rho$. In this regime nominal prices are pinned down according to the fiscal theory of the price level.*

Proof Consider an active fiscal stance, i.e., $\gamma < \rho$. In this regime the eigenvalue $\rho - \gamma$ is positive. Hence, monetary policy must bring about two stable eigenvalues via $\widehat{A}$, the upper-left 3×3 submatrix of A. Note Table A.2. Equilibrium is determinate only where $ind(\overline{x}) = 2$. A necessary condition for this case is $\det(\widehat{A}) > 0$. But this is not a sufficient condition, as it applies also for fixed points with $ind(\overline{x}) = 0$. We can rule out the case where $ind(\overline{x}) = 0$ by requiring that this monetary policy also induce $\mathrm{tr}(\widehat{A}) < 0$. To conclude, where fiscal policy is passive, we can ensure that $ind(\overline{x}) = 2$ by requiring that monetary policy should bring about $r_1 r_2 r_3 > 0$ and $r_1 + r_2 + r_3 < 0$. Note equations (A20)-(A21). Solving $\frac{\sigma \widetilde{c}^* (\nu + R^*)^2 (\rho+\delta)^2}{\alpha \nu \xi_k} (\alpha - 1) > 0, \rho + (\nu + R^*) \frac{\alpha - 1}{\alpha} + (\rho + \delta)(1 + \frac{1}{\nu} R^*) < 0$ we obtain that $\alpha < \frac{1}{1 + \frac{\rho}{\nu + R^*} + \frac{\rho + \delta}{\nu}} < 1$.

Consider a passive fiscal stance, i.e. $\gamma > \rho$. In this regime the eigenvalue $\rho - \gamma$ is negative. Hence, monetary policy must induce that $\widehat{A}$ has only one stable eigenvalue. A necessary condition for this case is $\det(\widehat{A}) < 0$, and to rule out the possibility that monetary policy induces three stable roots we require that $\mathrm{tr}(\widehat{A}) > 0$.

Solving $\frac{\sigma \widetilde{c}^* (\nu + R^*)^2 (\rho+\delta)^2}{\alpha \nu \xi_k} (\alpha - 1) < 0, \rho + (\nu + R^*) \frac{\alpha - 1}{\alpha} + (\rho + \delta)(1 + \frac{1}{\nu} R^*) > 0$ we obtain that $\alpha > 1$. We have thus verified that Leeper's (1991) result obtains in a production economy with finance constraints on investment and lump-sum taxation.

Proof of Proposition 8 The proof builds on:

I) a result that $\phi_\alpha \left[\phi_\beta + \phi_\gamma \left(\mathcal{RSF}^*{-}1\right) \frac{\tau^*}{1-\varphi^*}\right] < 0$;

II) a result that $\phi_\beta > 0$;

III) an assumption that $\mathcal{RSF}^*{<}1$

To prove the proposition one must show that there is a range $[1, \overline{\phi_\gamma})$ such that any $\phi_\gamma \in [1, \overline{\phi_\gamma})$ satisfies the condition $\phi_\beta + \phi_\gamma \left(\mathcal{RSF}^*{-}1\right) \frac{\tau^*}{1-\varphi^*} > 0 \Leftrightarrow \phi_\gamma < \frac{\phi_\beta}{(1-\mathcal{RSF}^*)\frac{\tau^*}{1-\varphi^*}} \Leftrightarrow \phi_\gamma < \frac{\beta + \frac{\tau^*}{1-\varphi^*}}{-\frac{1}{\nu}R^* + \frac{\tau^*\varphi^*}{1-\varphi^*}}$ (we obtain the right-hand side after substituting in the expressions for ϕ_β and $\mathcal{RSF}^*$)

Note that $\varphi^* < 1$ and $\beta, R^* > 0$. Thus the numerator at the right-hand side of the last inequality is greater than the denominator and the denominator is positive. As a result, the entire expression is positive and strictly greater than one. Let $\overline{\phi_\gamma} \equiv \frac{\beta + \frac{\tau^*}{1-\varphi^*}}{-\frac{1}{\nu}R^* + \frac{\tau^*\varphi^*}{1-\varphi^*}} > 1$.

So we have proved that any $1 \leq \phi_\gamma < \overline{\phi_\gamma}$ must interact with $\phi_\alpha < 0$. *QED.*

Proof of Proposition 9 Equilibrium is determinate only where $ind(\overline{x}) = 2$. $B_{[\gamma=0]}$ is block recursive with one positive eigenvalue at the lower right 1×1 submatrix, and so we obtain the dimension of the stable manifold only by examining $\widehat{B_1}$. Observe Table A.2. $\det(\widehat{B_1}) > 0$ is a necessary condition. Furthermore, we must rule out the case where $ind(\overline{x}) = 0$ by requiring that this policy also induce $\mathrm{tr}(\widehat{B_1}) < 0$. To conclude, we can ensure that $ind(\overline{x}) = 2$ by implementing a policy that brings about $r_1 r_2 r_3 > 0$ and $r_1 + r_2 + r_3 < 0$.

Note that $r_1 r_2 r_3 > 0 \Leftrightarrow \phi_\alpha \phi_\beta < 0$, and given that $\phi_\beta > 0$ we obtain that $\phi_\alpha < 0$ is a necessary condition for determinacy. $\phi_\alpha < -\frac{\rho + f_k^*}{\nu + R^*} < 0$ is sufficient to ensure determinacy because it induces both that $\det(\widehat{B_1}) > 0$ and that $\mathrm{tr}(\widehat{B_1}) < 0$, which rules out the possibility that $ind(\overline{x})$ is zero and verifies that it equals two. *QED.*

Proof of Proposition 10 The proof follows directly from the following theorems. Specifically, in the terminology of Theorems 1 and 2, we choose $\overline{y} = \overline{x}$ and $g(y)$ that differs from $f(x)$ up to the perturbation of γ.

Preliminaries

Theorem 1 [Hirsch and Smale (1976) Chap.16] *Let $f : W \to E$ be a C^1 vector field and $\overline{x} \in W$ an equilibrium of $\dot{x} = f(x)$ such that $Df(\overline{x}) \in L(E)$ is invertible. Then there exists a neighborhood $U \subset W$ of $\overline{x}$ and a neighborhood $\Re \subset \mho(W)$ of f such that for any $g \in \Re$ there is a unique equilibrium $\overline{y} \in U$ of $\dot{y} = g(y)$. Moreover, if E is normed, for any $\epsilon > 0$ we can choose $\Re$ so that $|\overline{y} - \overline{x}| < \epsilon$.*

Theorem 2 [Hirsch and Smale (1976) Chap.16] *Suppose that $\overline{x}$ is a hyperbolic equilibrium. In Theorem 1, then, $\Re$ and U can be chosen so that if $g \in \Re$, the unique equilibrium $\overline{y} \in U$ of $\dot{y} = g(y)$ is hyperbolic and has the same index as $\overline{x}$.*

Proof Consider now complex fiscal rules that exhibit $\gamma \neq 0$. In what follows I show that for small perturbations of γ near $\gamma = 0$ the system is structurally stable. Consider the system $\dot{x}_t = g_{[\gamma]}(x_t)$ where $\gamma = 0 + \varepsilon$, $\varepsilon > 0$. Then a linearization reads

$$\dot{x}_t = \left[B_{[\gamma=0]} + \varepsilon\Delta\right] \times (x_t - x^*) \tag{A22}$$

where

$$\Delta \equiv \begin{bmatrix} 0 & 0 & 0 & -\frac{\sigma(\rho+\delta)}{1-\tau^*}\frac{\widetilde{c}^*}{\widetilde{a}^*} \\ 0 & 0 & 0 & \frac{\nu}{\alpha}\frac{f_k^*}{f^*}\frac{1}{\widetilde{a}^*} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\widetilde{a}^*} \end{bmatrix}.$$

Since $B_{[\gamma=0]}$ is invertible, by the implicit function theorem, $\dot{x}_t = g_{[\gamma]}(x_t)$ continues to have a unique solution $x^{**} = x^* + O(\varepsilon)$ near x^* for sufficiently small ε. Moreover, since we restrict attention to a set of policies that satisfy Proposition 4, we ensure that $B_{[\gamma=0]} + \varepsilon\Delta$ is invertible, which implies that $x^{**} = x^*$ is the unique solution to equation (A22). Furthermore, since the matrix of the linearized system $B_{[\gamma=0]} + \varepsilon\Delta$ has eigenvalues that depend continuously on ε, no eigenvalues can cross the imaginary axis if ε remains small with respect to the magnitude of the real parts of the eigenvalues of $B_{[\gamma=0]}$. Thus, the perturbed system (A22) has a unique fixed point with eigenspaces and invariant manifolds of the same dimensions as those of the unperturbed system, and with an ε that is close in position and slope to the unperturbed manifolds. The main idea of this proposition is that perturbations are in the parameter space $\{\gamma\}$. By construction such perturbations do not change the steady state itself. However, they may change the phase portrait of the steady state. Thus, starting from a determinate equilibrium, as long as γ does not reach its bifurcation point, the phase portrait of the (unchanged) steady state should not be affected by the perturbation.

Preliminaries to Section 6 We add to the model TFP shocks and fiscal policy shocks. Let A_t, T_t denote levels of technology and lump-sum transfers, respectively. Then, assuming that the technology level in the economy and the level of government transfers follow AR(1) processes, A_t and T_t evolve according to:

$$dA_t = (\rho_A - 1)(A_t - A^*)\,dt + \varepsilon_{a,t}\sigma_a$$
$$dT_t = (\rho_T - 1)(T_t - T^*)\,dt + \varepsilon_{T,t}\sigma_T$$

where $\varepsilon_{a,t}, \varepsilon_{T,t}$ are shocks to technology and fiscal policy, respectively, and $\rho_A, \rho_T < 1$ denote their persistence. We also add shocks to monetary policy so the interest rate becomes
$R(\pi_t) = \rho + \pi^* + \alpha(\pi_t - \pi^*) + \widetilde{R_t}$
$d\widetilde{R_t} = (\rho_R - 1)\left(\widetilde{R_t} - 0\right) dt + \varepsilon_{R,t}\sigma_R$

where $\widetilde{R_t}$ represents an innovation to the interest rate rule, $\varepsilon_{R,t}$ is a shock, and $\rho_R < 1$ denotes its persistence. We assume that the monetary innovations follow an AR(1) process and decay to zero at a rate of $\rho_R - 1$. According to this specification, the deterministic part of impulse responses satisfy the following ODE system:

$$\begin{bmatrix} \dot{\widetilde{R_t}} \\ \dot{A_t} \\ \dot{T_t} \\ \dot{c_t} \\ \dot{\pi_t} \\ \dot{k_t} \\ \dot{a_t} \end{bmatrix} = \begin{bmatrix} \rho_R - 1 & 0 & 0 & & 0 & 0 & 0 & 0 \\ 0 & \rho_A - 1 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_T - 1 & & 0 & 0 & 0 & 0 \\ -\frac{\sigma(\rho+\delta)}{\nu+R^*} & \frac{\sigma\widetilde{c}^*(\rho+\delta)}{A^*(1-\tau^*)} f^*(1-\beta-\tau^*) & & 0 & & & & \\ (\rho+\delta+\nu+R^*)\frac{1}{\alpha} & -\frac{\nu}{\alpha}\frac{f_k^*}{A^*}(1-\beta-\tau^*) & & 0 & & B & & \\ 0 & \frac{f^*}{A^*} & & 0 & & & & \\ a^* - \frac{1}{\nu} f^* & -\frac{f^*}{A^*}\left(\frac{1}{\nu}R^* + \beta + \tau^*\right) & & 1 & & & & \end{bmatrix} \begin{bmatrix} \widetilde{R_t} - 0 \\ A_t - A^* \\ T_t - T^* \\ c_t - c^* \\ \pi_t - \pi^* \\ k_t - k^* \\ a_t - a^* \end{bmatrix}.$$

where B is the 4X4 matrix that is specified by eq. (A1). We can see that the new system is block-recursive. Since $A_t, T_t, \widetilde{R_t}$ are predetermined variables with stable eigenvalues [$(\rho_A - 1)$,$(\rho_T - 1)$, and $(\rho_R - 1)$ respectively], the stability properties of the new system are determined by the stability properties of matrix B.

Figure 4.1 enters here

Figure 4.2 enters here

Figures 4.1 and 4.2 show the responses of the model technology shocks, with a yearly autoregressive factor of 0.92. First, Figure 4 exhibits an important feature of the model that is consistent with what we know in modern macroeconomics: output and inflation co-move. Notice that under all regimes, a positive TFP shock generates persistent increases in both output and inflation, and a negative TFP shock generates persistent reductions in both inflation and output. Second, it seems that regimes that feature passive monetary policies, i.e. FTPL and UMA, emphasize output smoothing, whereas the Taylor-rule regime, which features an active monetary policy, emphasizes consumption smoothing. Finally, and most importantly, there are marked differences between the impulse responses of the model economy under UMA and FTPL. Both regimes, FTPL and UMA, bring about a similar time path of output. However, the mechanisms are different, as is evident from the time paths of debt and consumption. Under the FTPL regime, the negative productivity shock implies a fall in future output, and as a result a fall in tax revenues. Since tax rates are not expected to rise sufficiently to offset the fall in tax revenues, debt devalues under the FTPL regime. Debt devaluation has a negative wealth effect, and in consequence households reduce consumption. By contrast, under the UMA regime debt devaluation is not in operation, and as a result the wealth channel is mute on the impact of the shock. Instead, substitution effects come into play. The negative TFP shock in Figure 4.2 reduces the after-tax marginal product of capital. Although income tax and nominal interest both decline so as to offset the negative TFP shock, all in all the marginal product of capital declines. The effects of such adjustments come from the households' Euler equation: reductions in the after-tax marginal product of capital motivate the private sector to increase consumption.