A Fixed-Point Theory of Price Level Determination in General Equilibrium

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Abstract

This paper offers a fixed-point approach to the issue of price level determination in general equilibrium. It arrives at a solution method for rational expectations models with missing initial conditions for financial wealth. The paper emphasizes the calculation of an equilibrium initial valuation for government debt via a Krasnoselski-Mann-Bailey theorem. The analysis is performed from a global analysis perspective. Abstracting from policies that bring about zero eigenvalues permits us to draw conclusions about global dynamics. This approach builds on the Hartman-Grobman Theorem and implies no loss of generality.

JEL Codes: C62; C68; E60; H30; H60.

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1 Introduction

In conventional macroeconomics, where the price level is determined by balance between money supply and money demand, a unique solution is obtained only where there are as many linearly independent boundary conditions as there are linearly independent predetermined variables. This restriction runs the risk of drawing wrong conclusions when there are missing initial conditions. For example, in models of monetary economies households accumulate assets that are denominated in local currency. In this type of models the initial real value of financial wealth is not given. Instead, a boundary condition exhibits how should the real value of nominal assets evolve along an equilibrium trajectory. This paper characterizes a solution method for variants of the Blanchard and Kahn (1980) model with a significant extension that is typical to models with nominal assets.

Contribution: the main contribution of this paper is a theory that shows how the level of nominal prices is determined in general equilibrium. A transformation of ill conditioned boundary-value problems into well conditioned initial-value problem simplifies the analysis considerably. Two essential ingredients deliver this end: The first includes backward integration. It applies a time reversal to saddlepoint dynamics, which means it exploits the stability of an attractive manifold in a saddlepoint stable system.[1] Backward integration is conditional on having as many initial values as stable roots. However, where the state variables include non-indexed financial assets, at the outset

[1]This idea is discussed in detail in Buiter (1984) and Sims (2002).

only their nominal amount is given and not their real value. This brings us to the second ingredient which includes obtaining an initial valuation for nominal assets. This idea is discussed in Cochrane (2011) and Sims (2013) who emphasize that non-Ricardian fiscal policies are essential for the valuation of nominal assets. This idea goes back to the fiscal theory of the price level. The main result of my analysis is that the issue of price level determination corresponds to a solution of a fixed-point problem. This result arrives from plausible assumptions that bring about a condition that, in equilibrium, the real value of assets depends on the entire equilibrium trajectory. Hence, the real value of financial assets and the entire equilibrium trajectory must be obtained simultaneously.

I focus on production economies with finance constraints and distorting taxes. These assumptions add complications to the conventional model but have the merit of clarifying a few issues that the existing literature on the fiscal theory of the price level leaves obscure. We know that in equilibrium the real value of government debt equals the present value of the sum of future surpluses. However, where governments do not have access to lump-sum taxation, the time path of surpluses depends on the initial values of all forms of wealth, including government debt. Thus, computing the entire equilibrium involves a simultaneous determination of the initial valuation of government debt and the entire state space. This restriction indicates that the real value of government debt is a solution to a fixed point problem. I show that in our model, general equilibrium implies that the valuation of government debt is a fixed point of a continuous mapping that takes a closed interval of the real line into itself. This result has important implications: it is widely known that

Schauder's fixed-point theorem is a powerful method for proving existence theorems. If one wishes to prove that a given problem has a solution, he proceeds by associating it with a convex compact set E in some Banach space, and a continuous transformation T which carries E into itself. Schauder's theorem asserts that T must have at least one fixed-point in E. Developments attributed to Krasnoselski (1955), Mann (1953), and Bailey (1974) show that if E is a closed interval on the real line and T is continuous than the fixed point is unique in E and can be obtained as a limit to a sequence of points obtained via an iteration process. I implement this theory on the boundary to prove that the price level in a fiscal theoretic framework is uniquely determined. An important upshot of this theory is a simple algorithm for computing the initial value of government liabilities as well as the entire trajectory.

Related Literature: The canonical foundations of the fiscal theory of the price level are set in Sargent and Wallace (1981), Leeper (1991), Sims (1994), Woodford (1995), and Bassetto (2002). In a recent paper Cochrane (2011) argues that models with Taylor rules do not determine the price level any better than classic fixed interest rate targets. According to Cochrane (2011) price level determination in general equilibrium requires ingredients beyond the Taylor principle, such as a non-Ricaridan fiscal regime. This idea arrives from the fiscal theory of the price level where the initial (real) valuation of government liabilities is not given. Instead, a boundary condition exhibits how should the real value of government liabilities evolve along an equilibrium trajectory. Adjustments to the level of nominal prices are made so as to satisfy the boundary condition. Sims (2013) emphasizes that our thinking about inflation, monetary policy, and fiscal policy should be based on models of the this type. This paper is also related to research that assumes distortionary taxation and studies non-Ricardian fiscal policies. Prominent work in this literature includes Davig and Leeper (2010), Davig et. al.(2010, 2011), and Bi et. al. (2013). The solution method in this literature usually employs the monotone map method from Coleman (1991). This paper goes around complications[2] associated with the monotone map method as is imposes the transversality condition and ensures, by construction, the local uniqueness of a bounded rational expectations equilibrium.

The rest of the paper is organized as follows: section 2 describes the environment assuming a production economy, a liquidity constraint on all transactions, and a government that has access only to a distortionary taxation technology. The model nests monetary and fiscal feedback rules that are usually assumed in the underlying literature. Sections 3 and 4 discuss the determinants of equilibrium dynamics. In particular, section 3 characterizes a non-Ricardian fiscal regime in the context of the transversality condition. Section 4 provides necessary and sufficient conditions for local uniqueness of a rational expectations equilibrium. Section 5 applies the Krasnoselski-Mann-Bailey theorem to prove that in our model economy a non-Ricardian

[2] In a nutshell, this method emphasizes discretization of the state space around the non-stochastic steady state for each state variable, guessing an initial set of decision rules, substituting these rules into the household's first-order necessary conditions, and iterating the system until it converges at every point in the state space. This method does not ensure local uniqueness. Also, it computes a solution using only first-order necessary conditions, so it does not formally impose transversality conditions. As a result, a) to obtain evidence of local uniqueness, one needs to perturb the converged decision rules in various dimensions and check that the algorithm converges back to the same solution; and b) one should simulate the model and compute the average debt level at each date to obtain evidence that the expected value of debt does not asymptotically explode.

regime delivers a unique determination of the initial real value of financial wealth and, as a result, a unique determination of the level of nominal prices. Section 6 concludes the paper. All proofs are deferred to a technical appendix.

2 A Model with Distortionary Taxation

The model is formulated in continuous time to simplify the algebra and to obtain general results analytically. To economize on notations, we abstract from stochastic transitory shocks and focus on perfect foresight equilibria. This causes no loss of generality.[3]

2.1 The Households Sector

The economy is closed and populated by a continuum of identical infinitely long-lived households, with measure one. The representative household enjoys consumption and inelastically supplies its time endowment in the labor market, so its lifetime utility is given by

$$U_t = \int_t^{\infty} e^{-\rho s} u(c_s) ds \tag{1}$$

where $\rho > 0$ denotes the rate of time preference, c_s denotes consumption per capita, and $u(\cdot)$ is twice differentiable, strictly increasing, strictly concave,

[3] The model discusses rational expectations equilibria. It can accommodate transitory shocks to technology, preferences, and policy instruments by representing the relevant parameters as stochastic processes.

and satisfies the usual limit conditions. Production takes place in a competitive sector via a constant-returns-to-scale production technology $f(k_t)$ where k_t denotes per-capita capital which depreciates at a rate δ. Finally, $f(k_t)$ is concave and twice differentiable. Money enters the economy via a constraint on all transactions: let m_t denote the per-capita stock of money denominated in the consumption good, and let ν denote a money velocity, then a formal representation of the liquidity constraint is

$$c_t + I_t \leq \nu m_t \tag{2}$$

where I_t denotes per-capita investment. We assume that the government has access only to distortionary taxation and that deficits are financed via bond creation. As a consequence, the representative household's budget constraint becomes

$$c_t + I_t + \dot{b}_t + \dot{m}_t = (R_t - \pi_t)b_t - \pi_t m_t + (1 - \tau_t)f(k_t) + T_t \tag{3}$$

where $\tau_t \in [0, 1]$ is an income tax rate, b_t is a real measure of the stock of non-indexed government bonds, R_t is the nominal rate of interest, π_t is the rate of inflation, and T_t is a real lump-sum transfer. Capital accumulates according to

$$\dot{k}_t = I_t - \delta k_t. \tag{4}$$

Altogether, the household maximizes its lifetime utility given by (1) subject to the constraints (2)-(4), with a borrowing constraint such that $lim_{t\to\infty} a_t^H e^{-\int_0^t [R_s - \pi_s] ds} \geq$

0 where $a_t^H \equiv b_t + m_t$. Each household chooses sequences of $[(c_t, I_t, m_t)]_{t=0}^{+\infty}$ so as to maximize its lifetime utility, taking as given the initial stock of capital k_0, the initial stock of (real) financial wealth a_0^H, and the time path $[(\tau_t, T_t, R_t, \pi_t)]_{t=0}^{+\infty}$ which are all exogenous from the household's viewpoint. The necessary conditions for an interior maximum are

$$u'(c_t) = \lambda_t (1 + \frac{1}{\nu} R_t) \quad (5a)$$

$$\mu_t = u'(c_t) \quad (5b)$$

$$\zeta_t = \frac{1}{\nu} R_t \lambda_t \quad (5c)$$

$$\zeta_t (\nu m_t - c_t - I_t) = 0; \zeta_t \geq 0 \quad (5d)$$

where λ_t, μ_t are time-dependent co-state variables interpreted as the marginal valuations of financial wealth and capital, respectively; ζ_t is a multiplier associated with the liquidity constraint; and equation (5d) is a Kuhn-Tucker condition.

Restricting attention to positive nominal interest rates, equations (5c)-(5d) imply that ζ_t is positive, which in turn implies that the liquidity constraint is binding. Second, and after substituting $m_t = \frac{1}{\nu}(c_t + I_t)$ and $a_t^H = b_t + m_t$ into equation (3), the state and co-state variables must evolve according to

$$\dot{\lambda}_t = \lambda_t [\rho + \pi_t - R_t] \quad (6)$$

$$\dot{\mu}_t = -\lambda_t (1 - \tau_t) f'(k_t) + (\rho + \delta) \mu_t \quad (7)$$

$$\dot{k}_t = I_t - \delta k_t \quad (8)$$

$$\dot{a}_t^H = (R_t - \pi_t) a_t^H + (1 - \tau_t) f(k_t) + T_t - (c_t + I_t) \left(1 + \frac{1}{\nu} R_t\right) \quad (9)$$

Solving equation (9) yields that the household's intertemporal budget constraint is of the form

$$0 \leq lim_{t\to\infty} e^{-\int_0^t [R_s - \pi_s] ds} a_t^H =$$

$$a_0^H + \int_0^{\infty} e^{-\int_0^t [R_s - \pi_s] ds} \left[(1-\tau_t) f(k_t) + T_t - (c_t + I_t)\left(1 + \frac{1}{\nu} R_t\right)\right] dt$$

and the condition that its decisions are dynamically efficient yields the household's transversality condition

$$lim_{t\to\infty} a_t^H e^{-\int_0^t [R_s - \pi_s] ds} = 0. \tag{10}$$

Equations (6) – (10) fully describe the optimal program of a representative household for which $\left\{a_0^H, k_0, [(\tau_t, T_t, R_t, \pi_t)]_{t=0}^{+\infty}\right\}$ are exogenously given.

2.2 The Government

The government consists of a fiscal authority and a monetary authority. The consolidated government prints money, issues nominal bonds, collects taxes to the amount of $\tau_t y_t$ where y_t is output, and rebates to the households a real lump-sum transfer T_t. Its dollar-denominated budget constraint is therefore given by $R_t B_t + P_t T_t = \dot{M}_t + \dot{B}_t + P_t \tau_t y_t$, where P_t is the nominal price of a consumption bundle, $\dot{M}_t$ and $\dot{B}_t$ are net changes in the money and bond supply, respectively, and R_t is the nominal interest paid over outstanding debt. Dividing both sides of the nominal budget constraint by P_t and rearranging yields that government liabilities, denoted by $a_t^G \equiv \frac{M_t}{P_t} + \frac{B_t}{P_t}$, evolve according to

$$\dot{a}_t^G = \underbrace{\left(R_t - \pi_t\right) a_t^G}_{\text{interest payments on the debt}} - \underbrace{R_t m_t}_{\text{seigniorage}} + \underbrace{T_t - \tau_t y_t}_{\text{primary deficit}} \tag{11}$$

where $\pi_t \equiv \frac{\widehat{\dot{P}_t}}{P_t}$ and the hatted time derivative is a right derivative, referring to expected inflation. Equation (11) shows that since the consolidated budget is not necessarily balanced at every instant, deficits are financed via increments to government debt. As a result, government liabilities increase with the primary deficit and with the real interest paid over outstanding debt, and decrease with seigniorage.

Fiscal and Monetary Policies

We follow Leeper (1991) and consider policy rules that allow the scrutiny of first-order consequences of the time paths of nominal interest and income-tax rates. We assume that monetary policy follows an interest rate feedback rule,

$$R\left(\pi_t\right) = \rho + \pi^* + \alpha(\pi_t - \pi^*) \qquad where \qquad \alpha > 0 \tag{12}$$

Subsequent to Leeper (1991) monetary policies that exhibit $\alpha > 1$ are called active, and $\alpha < 1$ corresponds to passive policies. We assume an exogenous path for lump-sum transfers $T_t = \overline{T}$. Income-taxes follows rules that embed two features: first, there may be some automatic stabilizer component to movements in fiscal variables which is modeled as a contemporaneous response of the income-tax rate to deviations of output from the steady state; second, the income-tax rate is permitted to respond to the state of government debt. Altogether, the fiscal authority sets the income-tax rate according to

$$\tau(y_t, a_t^G) = \tau^* + \beta\frac{y_t - y^*}{y^*} + \gamma\frac{a_t^G - a^*}{a^*} \qquad where \qquad \beta, \gamma \geq 0 \tag{13}$$

and y^*, a^* are long-run output and a debt target, respectively.[4]

2.3 General Equilibrium

In equilibrium, a) the goods market clears

$$f(k_t) = y_t = c_t + I_t, \tag{14}$$

b) the money market clears

$$\frac{M_t}{P_t} = m_t = \frac{1}{\nu}\left(c_t + I_t\right), \tag{15}$$

and c) the assets market clears $a_t^G = a_t = a_t^H$.

Using the monetary policy rule and the fiscal rules, imposing market clearing

[4]There is much documented empirical relevance for fiscal rules that exhibit the features of eq. (13). See for example Bohn (1998), Leeper and Yang (2008) and Leeper, Plante, and Traum (2010). Recent contributions that emphasizes that tax rates may adjust to stabilize government debt include Schmitt-Grohe and Uribe (2007), Bi and Traum (2012), Bi, Leeper and Leith (2013), Fernandez-Villaverde et. al. (2013).

conditions, and assuming that the elasticity of intertemporal substitution in consumption is constant, we arrive at the following characterization of the general equilibrium of the economy:

Proposition 1 *In equilibrium with distortionary taxation and liquidity constraints, the aggregate dynamics satisfy the following ODE system:*

$$\frac{\dot{c}_t}{c_t} = \sigma \left\{ \left[\frac{1-\tau(f(k_t),a_t)}{1+\frac{1}{\nu}R(\pi_t)} f'(k_t) - \delta \right] - \rho \right\} \tag{16}$$

$$\dot{\pi}_t = \frac{\nu + R(\pi_t)}{\alpha} \left\{ [R(\pi_t) - \pi_t] - \left[\frac{1-\tau(f(k_t),a_t)}{1+\frac{1}{\nu}R(\pi_t)} f'(k_t) - \delta \right] \right\} \tag{17}$$

$$\dot{k}_t = f(k_t) - c_t - \delta k_t \tag{18}$$

$$\dot{a}_t = [R(\pi_t) - \pi_t] a_t + T_t - \left[\tau(f(k_t),a_t) + \frac{1}{\nu} R(\pi_t) \right] f(k_t). \tag{19}$$

Equation (16) is an Euler equation, where $\sigma > 0$ denotes the elasticity of intertemporal substitution in private consumption. In our economy the marginal product of capital is distorted by the income-tax and liquidity constraints. Notice that with no distortions equation (16) becomes the familiar Ramsey-type Euler equation. Equation (17) was obtained by taking a time derivative from the first-order condition ($5a$) and substituting in equation (6). It corresponds to a Fisher equation in which the nominal rate of interest varies with expected inflation and the real rate of interest. It shows that since capital and bonds are perfect substitutes at the private level, in equilibrium the distorted marginal product of capital net of depreciation must equal the real interest received from holding a risk-free bond minus the expected change in inflation after the policy response to inflation is internalized. Finally, equations (18)-(19) were obtained by substituting market clearing conditions (14)-(15) into equations (8)-(9). At this point we can characterize equilibrium where financial wealth is a state variable, however, only its nominal value is given:

Definition 1 *An equilibrium is a set of sequences* $\left\{[(c_t, \pi_t, k_t, a_t, \tau_t, T_t, R_t, P_t)]_{t=0}^{+\infty}\right\}$ *satisfying (16)-(19) given* $k_0 > 0$ and $A_0 \equiv M_0 + B_0 > 0$.

It follows from equation (16) that in a steady state,

$$f'(k^*) = (\rho + \delta)\frac{1 + \frac{1}{\nu}R^*}{1 - \tau^*}, \tag{20}$$

where τ^*, R^* denote the steady-state rates of the income-tax and nominal-interest. From equations (17) and (20), R^* must satisfy

$$R^* = \rho + \pi^* \tag{21}$$

where π^* is the steady-state rate of inflation. Equation (18) implies that the steady-state consumption is

$$c^* = f(k^*) - \delta k^* \tag{22}$$

Finally, equation (19) shows that in a steady-state equilibrium, government liabilities must satisfy

$$a^* = \frac{1}{\rho}\left[f(k^*)(\tau^* + \frac{1}{\nu}R^*) - \overline{T}\right] \tag{23}$$

Note that an equilibrium trajectory $[(c_t, \pi_t, k_t, a_t, \tau_t, T_t, R_t, P_t)]_{t=0}^{+\infty}$ should converge to the steady state $(c^*, \pi^*, k^*, a^*, \tau^*, T^*, R^*, P^*)$, whereas the system of equations (16) - (19) determines only four dimensions of the steady state. It is thus apparent at the outset that our model may exhibit global indeterminacy. Also, note that if the time path of inflation, $[(\pi_t)]_{t=0}^{+\infty}$, is unique and P_0 is determined, then the time path of nominal prices is also determined. This brings us to the following definition:

Definition 2 (Global Determinacy) *The equilibrium displays global determinacy if the system (16) - (19) has:*

a) A unique stationary solution $(c^*, \pi^*, k^*, a^*, \tau^*, T^*, R^*)$, *and*

b) a unique initial price level P_0, *and*

c) a unique equilibrium trajectory $[(c_t, \pi_t, k_t, a_t, \tau_t, T_t, R_t)]_{t=0}^{+\infty}$ *that converges to* $(c^*, \pi^*, k^*, a^*, \tau^*, T^*, R^*)$.

This leads us to the following proposition:

Proposition 2 *A necessary condition for global determinacy is that the government proclaims three 'exogenous' targets.*

Proposition 2 implies that the steady state is sustained only if the revenues from taxes and seigniorage equal the sum of transfers and debt service. Thus, as equation (23) links $\pi^*, \tau^*, a^*, \overline{T}$ to a balanced budget condition, three targets should be specified 'exogenously', and the fourth is implied by the stipulation to run a balanced budget in the steady state. For example, where the government proclaims the 'exogenous target' (π^*, τ^*, a^*) , the long-run surplus is implied according to $\overline{T} = \left[\tau^* + \frac{1}{\nu}R^*\right] f(k^*) - \rho a^*$. This result follows directly from (23). In the rest of the paper we assume that the government chooses (π^*, τ^*, a^*) as 'exogenous targets'. Note that proclaiming three targets is necessary for global determinacy, however this in not sufficient. In order to induce global determinacy the government should also take actions that ensure a unique determination of P_0 and a unique determination of the equilibrium trajectory. These issues receive attention in the rest of the paper.

3 Non-Ricardian Regimes and the Transversality Condition

In the following sections we pursue regimes that bring about a unique bounded rational expectations equilibrium and a unique determination of P_0. To characterize equilibrium correctly we must impose the condition that the household's intertemporal budget constraint holds with equality. Note that the choice of γ determines the rate of growth of government debt. In order to study the effect of γ on the evolution of government debt, we substitute the fiscal rule (13) into (19) and obtain that government debt evolves according to

$$\dot{a}_t = \left[R(\pi_t) - \pi_t - \gamma \frac{f(k_t)}{a^*}\right] a_t - f(k_t) \left[\tau^* + \beta \frac{f(k_t) - f(k^*)}{f(k^*)} - \gamma + \frac{1}{\nu} R(\pi_t)\right] + \overline{T}. \tag{24}$$

Solving equation (24) for a_t and letting $t \to \infty$, we arrive at

$$\lim_{t\to\infty} Q_t a_t = a_0 - \int_0^\infty Q_s X_s ds \tag{25}$$

where $Q_t \equiv e^{-\int_0^t \left[(R(\pi_s)-\pi_s)-\gamma \frac{f(k_s)}{a^*}\right] ds}$, $X_s \equiv \left\{\left[\tau^* + \beta \frac{f(k_s)-f(k^*)}{f(k^*)} - \gamma + \frac{1}{\nu} R(\pi_s)\right] f(k_s) - \overline{T}\right\}$.

X_s is the surplus at instant s and Q_s is its respective discount factor. As we assume that transfers are constant, the surplus flow has two components. The first component includes revenues from taxing all sources of income in the economy. Note that the tax rate includes an automatic stabilizer component, which we modeled as a response of the income-tax rate to deviations of output from the steady state. The second component of the surplus flow includes seigniorage revenues accruing from the constraint that all transactions in the economy are carried out in exchange for money. Also note that the discount factor has two components. The first stems from monetary policy and equals the real rate of interest, while the second stems from fiscal policy and attaches a growth premium to the surplus flow. Equation (25) has several implications. As we know, the left-hand side of the equation must equal zero in equilibrium. If γ is large enough, real debt will shrink back to its long-run level and the transversality condition is ensured. By contrast, if γ is too small it may bring an impression that the government lets its debt grow too fast. In what follows we discuss the effects of fiscal policy on the transversality condition. We start by defining Ricardian and non-Ricardian policy regimes

Definition 3 *Fiscal policy is considered Ricardian if the exogenous sequences and feedback rules that specify the policy regime imply that the transversality condition necessarily holds for any initial level of government debt. Fiscal policy is considered non-Ricardian if for any sequence* $[(c_t, \pi_t, k_t, a_t, \tau_t, T_t, R_t)]_{t=0}^{+\infty}$ *satisfying (16)-(19) there exists a unique* *valuation for government debt a(0) > 0 that is consistent with equilibrium.*

Notice eq. (25) and consider fiscal rules that exhibit $\gamma \geq \rho\widetilde{a}^{*}$. Under this type of policies both sides of eq. (25) may grow to infinity since Q_t is not contracting. Note, however, that eq. (25) reads

$$\lim_{t\to\infty} e^{-\int_0^t [R(\pi_s)-\pi_s]ds} a_t = \lim_{t\to\infty} \left[\frac{a_0 - \int_0^t Q_s X_s ds}{e^{\int_0^t \gamma \frac{f(k_s)}{a^*} ds}} \right]. \tag{26}$$

In this formulation one can see that $\gamma \geq \rho\widetilde{a}^{*}$ implies that both the numerator and the denominator of the right-hand side of eq. (26) expand to infinity. It is straightforward to show via L'Hospital's law that where $\gamma \geq \rho\widetilde{a}^{*}$, the limit of the expression on the right-hand side of eq. (26) is zero. Thus, policy rules that exhibit $\gamma \geq \rho\widetilde{a}^{*}$ ensure that the household's transversality condition is not violated and should therefor be considered as "Ricardian". However, under such rules the initial level of nominal prices remains indeterminate [see Woodford (1995) and Cochrane (2001, 2011) for an extensive discussion on this issue].

Consider now fiscal rules that exhibit $\gamma < \rho\widetilde{a}^{*}$, then

Proposition 4 *Fiscal rules are non-Ricardian if and only if* $\gamma < \rho\widetilde{a}^{*}$. *In this case the initial real value of government debt,* a_0*, must jump so as to satisfy* $a_0 = \int_0^\infty Q_s X_s ds.$

Where $\gamma < \rho\widetilde{a}^*$, Q_t is contracting and the value $\int_0^\infty Q_s X_s ds$ is not necessarily equal to a_0. In such cases, the right-hand side of eq. (25) is non-zero, which implies that the transversality condition does not hold. In this case solvency is brought about only via changes to a_0 so as to equate the right-hand side of eq. (25) to zero. Thus, non-Ricardian fiscal rules are necessary for obtaining an initial valuation to government debt, a_0. Since $A_0 \equiv M_0 + B_0$ is given, Proposition 4 also shows how to pin point the initial price level P_0. Note however that a_0 depends on the time path of $[(Q_s, X_s)]_{t=0}^{+\infty}$. Where taxes are lump sum and future surpluses are independent of real allocation, it is possible to pinpoint the initial value of government debt based on the surpluses alone, and whether there is a unique trajectory or multiplicity of trajectories of real allocations has no relevance to our ability to calculate the present value of future surpluses. In contrast, where the government has access only to distortionary taxation, tax revenues become a feature of equilibrium. Moreover, in our model output and the tax rate are determined simultaneously in equilibrium. Hence, future surpluses depend on the entire equilibrium trajectory and, as a result, so does a_0.

4 The Stable Manifold

We know by now that a_0 is a projection of equilibrium. Thus, a unique determination of the equilibrium trajectory is a prerequisite for a unique determination of a_0. Let $\dot{x}_t = g(x_t)$, $x_t' \equiv (c_t, \pi_t, k_t, a_t)'$ denote the non-linear system of equations (16)-(19). Then a linear approximation near the steady state reads

$$\dot{x_t} = B \times (x_t - x^*) \tag{27}$$

where x^* is a steady state equilibrium. When B has no eigenvalues with zero real parts, the steady state x^* is a hyperbolic fixed point for which the Hartman-Grobman Theorem and the Stable Manifold Theorem for non-linear systems hold. As a result, the asymptotic behavior of solutions near x^* – and hence its stability type – are determined by the linearization (27). Let $\varphi_{(\tau_t, y_t)} \equiv \frac{\partial \ln(\tau_t y_t)}{\partial \ln(\tau_t)} = 1 + \frac{\partial \ln(y_t)}{\partial \ln(\tau_t)}$ denote the marginal revenue generated from an increase in taxes, and one can interpret $\varphi_{(\tau_t, y_t)}$ as the slope of the income-tax Laffer curve. Note that the second term is negative as higher taxes decrease output, so the elasticity of tax revenue with respect to tax rates is less than one. In this economy $y_t = f(k_t)$, accordingly $\varphi_{(\tau_t, y_t)} = 1 + \frac{\tau_t}{f(k_t)} \frac{\partial f(k_t)}{\partial \tau_t} = 1 + \frac{\tau_t}{f(k_t)} f'(k_t) \frac{dk_t}{d\tau_t}$. It is straightforward to obtain[5] that $\frac{dk^*}{d\tau^*} = \frac{1}{1-\tau^*} \frac{f'(k^*)}{f''(k^*)}$, and therefore the slope of the income-tax Laffer curve near the steady state reads

$$\varphi_{(\tau^*)} = 1 + \frac{\tau^*}{1 - \tau^*} \frac{[f'(k^*)]^2}{f(k^*) f''(k^*)} \tag{28}$$

and we obtain analytically that the product of system's eigenvalues equals[6]
$- [\widetilde{c}^* \nu \sigma \rho f_k^{*2}] \frac{\alpha - 1}{\alpha} \left[\beta + \frac{\tau^*}{1-\varphi^*} + \frac{\gamma}{\rho \tilde{a}^*} (\frac{1}{\nu} R^* - \frac{\tau^* \varphi^*}{1-\varphi^*}) \right]$
At this point additional terminology is required so as to characterize the channels through which monetary and fiscal policies affect local dynamics.

[5] By applying the implicit function theorem on equation (20).
[6] See the preliminary section in the technical appendix.

The slope $\varphi_{(\tau^*)}$ is related to the degree to which a tax cut is self-financing, defined as the ratio of additional tax revenues due to general equilibrium effects and the lost tax revenues due to the tax cut. Formally, the degree to which a tax cut is self-financing, denoted by $\mathcal{RSF}$, is calculated as $\mathcal{RSF} = 1 - \frac{1}{f(k_t)} \frac{d\left[f(k_t)(\tau_t + \frac{1}{\nu} R_t)\right]}{d\tau_t}$ where the denominator add up all tax revenues. Simple algebra yields that the rate of self-financing near the steady state depends on the elasticity of tax revenues, the tax-rate target, and the inflation target, and reads

$$\mathcal{RSF}^* = 1 - \frac{\varphi^* - 1}{\tau^*} \left[\frac{1}{\nu} R^* - \frac{\tau^* \varphi^*}{1 - \varphi^*}\right] \tag{29}$$

and we restrict our attention from now on to economies where the government chooses an income-tax target, τ^*, such that its respective $\mathcal{RSF}^*$ is not greater than one. Also, let $\phi_\alpha \equiv \frac{\alpha - 1}{\alpha}, \phi_\beta \equiv \beta + \frac{\tau^*}{1 - \varphi^*}, \phi_\gamma \equiv \frac{\gamma}{\rho \tilde{a}^*}$, denote the net effects of government actions on the real interest rate, output, and tax revenues, respectively.

One can think of ϕ_α as the net effect of the monetary response on the real rate of interest when inflation is above target. A negative ϕ_α implies that the monetary authority lets the real rate of interest drop below its long-run level when inflation is above target.

We interpret ϕ_γ as the net effect of tax hikes on the secondary deficit. $\phi_\gamma < 1$ implies that when debt increases above its long-run level, income taxes may rise. However, the increase in tax revenues is not enough to cover the increase in interest payments and as a result cannot stop debt from growing. According to Proposition 4, such rules are non-Ricardian.

Finally, we interpret ϕ_β as the effect of fiscal policy on the net marginal product of capital. Note that $\frac{d\ln\left[\frac{1-\tau(k_t,a_t)}{1+\frac{1}{\nu}R(\pi_t)}f'(k_t)\right]}{d\ln f(k_t)} \approx -\frac{1}{1-\tau^*}\phi_\beta$. That is, the evolution of after-tax marginal product of capital along an equilibrium trajectory is sensitive to responses of the fiscal policy to output. When fiscal policy exhibits responses to output such that $\phi_\beta < 0$, the after-tax marginal product of capital becomes positively associated with output and such policies induce multiple equilibria. The intuition is as follows: start from a steady state equilibrium, and suppose that the future return on capital is expected to increase. Indeterminacy cannot occur without distorting taxes, since a higher capital stock is associated with a lower rate of return under constant returns to scale. However, a feedback income-tax rule that exhibits $\phi_\beta < 0$ causes the after-tax return on capital to rise even further, thus validating agents' expectations, and any such trajectory is consistent with equilibrium. By contrast, a fiscal stance $\phi_\beta > 0$ reduces higher anticipated returns on capital from belief-driven expansions, thus preventing expectations from becoming self-fulfilling.[7] We therefore restrict attention from now on to policies that exhibit $\phi_\beta > 0$.

Altogether, we arrive at the following characterization of the eigenvalues of the system (16)-(19):

[7] This intuition explains the indeterminacy that arises in the models of Schmitt-Grohe and Uribe (1997) and Guo and Harrison (2004) with distortionary taxes.

$$r_1 r_2 r_3 r_4 = -\kappa\phi_\alpha \left[\phi_\beta + \phi_\gamma \left(\mathcal{RSF}^* - 1\right) \frac{\tau^*}{1-\varphi^*}\right] \tag{30}$$

$$r_1 + r_2 + r_3 + r_4 = 2\rho + (\nu + R^*)\phi_\alpha + f_k^* - \rho\phi_\gamma \tag{31}$$

where $r_i \;\; i = 1, .., 4$ denote the eigenvalues of B and $\kappa \equiv \widetilde{c}^* \nu\sigma\rho f_k^{*2} > 0$ is a constant. Having this, and given that (k_t, a_t) are predetermined, Proposition 5 follows directly from equation (30):

Proposition 5 *The steady state x^* is hyperbolic if and only if $\phi_\alpha \left[\phi_\beta + \phi_\gamma \left(\mathcal{RSF}^* - 1\right) \frac{\tau^*}{1-\varphi^*}\right] \neq 0$. Furthermore, a necessary condition for the existence of a unique bounded rational expectations equilibrium is that*

$$\phi_\beta > 0 \tag{32}$$

and

$$\phi_\alpha \left[\phi_\beta + \phi_\gamma \left(\mathcal{RSF}^* - 1\right) \frac{\tau^*}{1-\varphi^*}\right] < 0$$

Proposition 5 reveals that the interaction between the monetary policy and the fiscal policy must bring about one stable root and one unstable root. The intuition is the following: consider a one sector growth model with no distortions. In this case saddle-path stability is ensured by intertemporal substitution and the existence of one predetermined variable - the stock of productive capital. Adding tax and liquidity distortions to the economy augments the model to the system (16)-(19) and adds one predetermined variable [real debt]. As a result, in order to display saddle-path stability the entire system must have two stable roots and two unstable roots. Since one stable root is ensured by intertemporal substitution, saddle-path stability can be achieved only if the combination of monetary and fiscal policies can add to the system one stable root and one unstable root. Note that we focus on price level determinacy and therefore restrict attention to non-Ricardian fiscal policies. Such fiscal policies exhibit $\phi_\beta > 0$ and $\phi_\gamma < 1$ which means that the fiscal rule drives the economy away from the steady state at least in two dimensions: a) it reduces higher anticipated returns on capital, and b) it does not stop government debt from growing. Since a non-Ricaridan fiscal rule seems to induce an unstable root we can conclude that monetary policy must induce a stable root in order to bring about saddle-path stability. This conjecture arrives from Proposition 5 and the following Lemma:

Lemma 1 *All non-Ricardian fiscal rules satisfy* $\phi_\beta + \phi_\gamma \left(\mathcal{RSF}^* - 1\right) \frac{\tau^*}{1-\varphi^*} > 0$

The bottom line of Proposition 5 and Lemma 1 is that a non-Ricardian fiscal rule must interact with a passive monetary rule. We define from now on a non-Ricardian regime as an interaction between fiscal and monetary rules according to the following characteristics:

Definition 4 *A non-Ricaridan regime is an interaction between a non-Ricardian fiscal rule and a passive monetary rule, i.e. an interaction between monetary policies that exhibit* $\phi_\alpha < 0$ *and fiscal policies that exhibit* $\phi_\beta > 0$ *and* $\phi_\gamma < 1$.

and we know from Proposition 5 and Lemma 1 that non-Ricardian regimes satisfy the necessary condition for uniqueness. In order to obtain sufficient conditions, we focus on a baseline regime for which it is possible to obtain sufficient conditions. We then perturb the baseline regime so as to approximate the general case.[8] Consider a fiscal rule that exhibits $\phi_\gamma = 0$. In this regime, the income-tax rate does not respond to deviations of debt from its long-run level. Substituting $\gamma = 0$ into eq. (27), the system becomes

$$\dot{x_t} = B_{[\gamma=0]} \times (x_t - x^*) \tag{33}$$

where

$$B_{[\gamma=0]} = \begin{bmatrix} \widehat{B_1} & \varnothing \\ \widehat{B_2} & \rho \end{bmatrix}$$

and $\widehat{B_1}$ is the upper left 3×3 submatrix of B, $\widehat{B_2}$ is the 1×3 vector $(B_{4,1}, B_{4,2}, B_{4,3})$, and $\varnothing$ is a 3×1 vector of zeros. Examining $B_{[\gamma=0]}$, the dynamics of (c, π, k) are independent of government liabilities. This feature has two implications: (a) one eigenvalue of the (c, π, k, a) system is $\rho > 0$; (b) the remaining three eigenvalues are determined by $\widehat{B_1}$ so that the dynamics of (c, π, k) are completely determined by $\widehat{B_1}$. It is straightforward to show that the three remaining eigenvalues satisfy

$$\begin{aligned} r_1 r_2 r_3 &= -\kappa \phi_\alpha \phi_\beta && (34) \\ r_1 + r_2 + r_3 &= \rho + (\nu + R^*)\phi_\alpha + f_k^* && (35) \end{aligned}$$

[8] This approach is very useful in differential analysis. See Hirsch and Smale (1976) and Guckenhaimer and Holmes (1983).

which leads us to the following proposition:

Proposition 6 Where $\phi_\beta > 0$ *and fiscal policy targets only output, a sufficient condition for the existence of a unique bounded rational expectations equilibrium is that monetary policy exhibits* $\phi_\alpha < -\frac{f_k^* + \rho}{\nu + R^*} < 0$.

Lemma 2 *Consider a regime* $\{(\phi_{\overline{\alpha}}, \phi_{\overline{\beta}}, \phi_{\overline{\gamma}}) \mid \phi_{\overline{\alpha}} < 0, \phi_{\overline{\beta}} > 0, \phi_{\overline{\gamma}} = 0\}$ *that brings about a unique bounded rational expectations equilibrium near a hyperbolic steady state* x^*. *Then, perturbations to* ϕ_γ *in the neighborhood of* $\phi_{\overline{\gamma}} = 0$ *do not change the phase portrait of* x^* *as long as* ϕ_γ *is not perturbed until its bifurcation point. Specifically, given* $\phi_{\overline{\alpha}} < 0$ *and* $\phi_{\overline{\beta}} > 0$, *any regime that exhibits* $\phi_\gamma > 0$ *will also induce a unique bounded rational expectations equilibrium as long as the multiple of eigenvalues in the perturbed system does not change signs.*

Proposition 7 *Any non-Ricardian regime induces a unique bounded rational expectations equilibrium.*

Proposition 6 argues that near a hyperbolic steady state x^* where fiscal policy targets only output a passive monetary rule induces a unique equilibrium path. Lemma 2 argues that small perturbations to the fiscal rule near x^* can preserve equilibrium uniqueness. Proposition 7 argues that any non-Ricardian regime can be viewed as a small perturbation to the fiscal rule near $\{(\phi_{\overline{\alpha}}, \phi_{\overline{\beta}}, \phi_{\overline{\gamma}}) \mid \phi_{\overline{\alpha}} < 0, \phi_{\overline{\beta}} > 0, \phi_{\overline{\gamma}} = 0\}$ and therefore preserves the local uniqueness of the baseline regime in Proposition 6.

5 The price level as a solution to a fixed point problem

This section outlines a fixed-point theory for a unique determination of nominal prices in general equilibrium. The theory consists of three central ideas. The first idea has been introduced by Mulligan and Sala-i-Martin (1991, 1993) and includes the transformation of an inherently unstable boundary value problem into a stable initial value problem which can then be solved easily using standard methods. However, we do not eliminate but reverse time so as to exploit the stability of backward looking systems. The second idea is that an approximation of the infinite time horizon is endogenously determined. It depends on the initial deviation of the backward looking system from its steady state.[9] The third idea is that we must make sure that our approach delivers a unique price level. We know by now that distortionary taxation implies that the real value of government debt and the equilibrium trajectory must be determined simultaneously. The Krasnoselski-Mann-Bailey theorem proves that in our model-economy any non-Ricardian regime delivers a unique determination of the initial real value of financial wealth and, as a result, a unique determination of the entire equilibrium trajectory. Let C be a closed convex subset of a Banach space E. A mapping T on C is called a non-expanding mapping if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $\{\overline{x}\}$ the set of fixed points of T. If C is compact, then Schauder's

[9]See for example Buiter (1984) and Sims (2002). This approach is very different from a backward shooting approach because shooting always implies the possibility of missing the target while backward integration obtains the solution from a given initial value to a given steady state up to a predetermined error at the first shot.

fixed-point theorem yields that $\{\overline{x}\}$ is non-empty. Obtaining members of $\{\overline{x}\}$ is then feasible via a Krasnoselski sequence. Krasnoselski's Theorem is the following result:

Krasnoselski's Theorem If C is a convex, bounded subset of a uniformly convex Banach space and if T is a mapping of C into a compact subset of C such that $\|T(x) - T(y)\| \leq \|x - y\|$, then the sequence obtained by choosing x_1 in C and defining $x_{n+1} = \frac{1}{2}[x_n + T(x_n)]$ converges to some z in C and $T(z) = z$.

Mann's (1953) theorem states that if T is a continuous function, not necessarily non-expanding, that takes a closed interval of the real line, $[a, b]$, into itself and has a unique fixed point, z in $[a, b]$, then there is a sequence that converges to z for all choices of x_1 in $[a, b]$. Mann's Theorem is the following:

Mann's Theorem If $T(x)$ is a continuous function carrying the interval $[a, b]$ into itself and having a unique fixed-point, z, on $[a, b]$, then (x_1, Λ, T) converges to z for all choices of x_1 on $[a, b]$ where (x_1, Λ, T) denotes the process of starting with an arbitrary point $x_1$in $[a, b]$ and applying the formulas $x_{n+1} = T(v_n)$ and $v_n = \frac{1}{n}\sum_{i=1}^{n} x_i$

Bailey (1974) offered a proof for the case in which C is a closed interval of the real line and T is a non-expanding mapping. Bailey's Theorem is the following:

Bailey's Theorem If T takes $[a, b]$ into itself and $\|T(x) - T(y)\| \leq \|x - y\|$, then the sequence obtained by choosing any x_1 in $[a, b]$ and defining $x_{n+1} = \frac{1}{2}[x_n + T(x_n)]$ converges to some z in $[a, b]$ and $T(z) = z$.

With reference to our model-economy, Mann and Bailey's theorems imply that if T is a continuous non-expanding function, that takes a closed interval of the real line, $[a, b]$, into itself and has a unique fixed point, z in $[a, b]$, then a Bailey's sequence converges to z for all choices of x_1 in $[a, b]$. This brings us to the point where we can calculate equilibrium valuations of government debt under non-Ricardian regimes. We recognize that given the initial stock of capital, the monetary rule, and the fiscal rule, the present value of future surpluses becomes a function of the valuation of government debt. The latter measure thus becomes a mapping from a closed interval on the real line to itself. Thus, it is straightforward to obtain the initial value of government debt as a limit to a Krasnoselski-Mann-Bailey sequence. Corresponding to the autonomous system $\dot{x}_t = g(x_t)$, specified in Proposition 1 and where $g : W \rightarrow E$, there are maps $\Phi : \Omega \rightarrow W$ where $(t, c, \pi, k, a, \tau, T, R) \in \Omega$ satisfying $\dot{\Phi}_t = g(\Phi_t)$. In view of definition 1, Φ_t is defined by letting $\Phi_t(c, \pi, k, a, \tau, T, R) \equiv \Phi(t, c, \pi, k, a, \tau, T, R)$ be a solution curve sending 0 to (k_0, a_0) and sending $+\infty$ to $(c^*, \pi^*, k^*, a^*, \tau^*, T^*, R^*)$. Propositions 6 and 7 provide conditions that guarantee the existence of a unique homeomorphism Φ_t on $[0, +\infty)$.

Let $\Psi \equiv \int_0^\infty Q_s X_s ds$ indicate the present value of future primary surpluses. Then $\Psi : W \rightarrow \Re_+$ corresponds to the mapping of the flow Φ_t into a positive real number. Let $\digamma$ denote the composition $\Psi \circ \Phi$. Then $\digamma : \Omega \rightarrow \Re_+$, and $\widetilde{\digamma}(a_t) \equiv \digamma(c_t, \pi_t, k_t, a_t, \tau^*, T^*, R^*)$ returns the valuation of government liabilities given that the tax rate target is τ^*, the inflation target is π^*, the debt target is a^*, the state variables are (a_t, k_t), and (c_t, π_t) are jump variables. In particular, $\widetilde{\digamma} : \Re_+ \rightarrow \Re_+$ is a single-dimensional function that measures solely the effect of the state variable a_t on the valuation of government debt, taking as given the stock of capital k_t and the 'exogenous' targets. The economy is in equilibrium if and only if $\forall t \in [0, +\infty)$ $a_t = \widetilde{F}(a_t)$. For example, we can say that we pinned down the initial value of financial wealth if we obtain an a_0 that is a solution to $a_0 = \widetilde{F}(a_0)$. Similarly, the economy is in it's steady state when $a^* = \widetilde{F}(a^*) = \frac{1}{\rho}\left[f(k^*)(\tau^* + \frac{1}{\nu}R^*) - \overline{T}\right]$. A crucial feature of $\widetilde{F}$ stems from the requirement that the economy resides in a non-Ricardian regime. In such regimes, increments to the real value of government debt are met by tax actions that do not fully offset the increase in government debt, and therefore cause a less than one-for-one increase in the present value of future surpluses. The formal representation of this feature is $\frac{\partial \widetilde{F}}{\partial a} < 1$, and this leads us to the following result:

Proposition 8 *Under non-Ricardian regimes* $\widetilde{F}(a)$ *is a non-expanding mapping.*

Proof A Taylor expansion for $\widetilde{F}(a)$ reads $d\widetilde{F}(a) = \left[\frac{\partial F}{\partial a}\right]_{a^*} da + residual.$

Taking norms yields that

$\left\|d\widetilde{F}(a)\right\| \leqslant \left\|\left[\frac{\partial F}{\partial a}\right]_{a^*}\right\| \|da\| + \|residual\|$

$\Rightarrow \left\|d\widetilde{F}(a)\right\| \leqslant \left\|\left[\frac{\partial F}{\partial a}\right]_{a^*}\right\| \|da\|$

and since non-Ricardian regimes imply that $\frac{\partial \widetilde{F}}{\partial a} < 1$ we obtain that $\left\|d\widetilde{F}(a)\right\| \leqslant \|da\|$. *QED.*

Once we establish that $\widetilde{F}(a)$ is a non-expanding mapping that maps a closed interval on the real line into itself, we apply a Krasnoselski-Mann-Bailey sequence to obtain a solution to the fixed -point problem $a_0 = \widetilde{F}(a_0)$. This concludes the theory laid out in this paper an yields that

Corollary *Any non-Ricardian regime that responds to three exogenous targets ensures global determinacy.*

The corollary summarizes the upshots of Propositions 1- 8. We were thus able to show that given $k_0 > 0$ and $A_0 \equiv M_0 + B_0 > 0$, the combination of a non-Ricardian regime and three explicit policy targets are sufficient to pin point the steady state, to induce the initial valuation of non-capital wealth, and to induce a unique sequence $[(c_t, \pi_t, k_t, a_t, \tau_t, T_t, R_t, P_t)]_{t=0}^{+\infty}$ that satisfies the aggregate dynamics (16)-(19) where $a_0 = \frac{A_0}{P_0}$ and converges to the steady state $(c^*, \pi^*, k^*, a^*, \tau^*, T^*, R^*)$ that the government wishes to achieve.

6 Concluding Remarks

This paper formalizes the consequences of combined changes in the level of income taxes and the nominal rate of interest designed so as to achieve long-run levels of public-debt, inflation, and output. We augment the Ramsey model to include distortionary taxation and liquidity constraints. As a result, changes in taxes influence output. We construct a dynamic setup that gets around complications associated with dynamic Laffer curves. In this setup, tax cuts are self-financing at rates higher than 1 minus the slope of the income-tax Laffer curve because of a wedge brought about by the monetary system.

The analysis focuses on two issues: a) the paper arrives at a solution method for rational expectations models that can be represented by systems of first order differential equations. This method is a continuous-time variant of the method of Blanchard and Kahn with missing initial condition for financial wealth. The paper emphasizes the calculation of an equilibrium initial valuation for financial wealth via a Krasnoselski-Mann-Bailey method. This result is innovative as it takes a fixed-point approach to the issue of price level determination in general equilibrium. b) the analysis is performed from a global analysis perspective. We abstract from policies that bring about zero eigenvalue. This approach, that builds on the Hartman-Grobman Theorem, permits us to draw conclusions about the global system from the linearized system, with no loss of generality.

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Technical Appendix

Preliminaries

Definition (the index of a fixed point) [Hirsch and Smale (1976)] *Let $\overline{x} \in \mathbb{R}^n$ be a hyperbolic equilibrium, that is, the eigenvalues of $Df(\overline{x})$ have nonzero real parts. In this case, the index $ind(\overline{x})$ of $\overline{x}$ is the number of eigenvalues (counting multiplicities) of $Df(\overline{x})$ having negative real parts.*

The Stable Manifold Theorem [Guckenhaimer and Holmes (1983) Theorem 1.3.2] *Suppose that $\dot{x} = f(x)$ has hyperbolic fixed point $\overline{x}$. Then there exist local stable and unstable manifolds $W^s_{loc}(\overline{x}), W^u_{loc}(\overline{x})$, of the same dimensions n_s, n_u as those of the eigenspaces E^s, E^u of the linearized system, respectively, and tangent to E^s, E^u at $\overline{x}$. $W^s_{loc}(\overline{x}), W^u_{loc}(\overline{x})$ are as smooth as the function f.*

The Hartman-Grobman Theorem [Guckenhaimer and Holmes (1983) Theorem 1.3.1] *If $Df(\overline{x})$ has no zero or purely imaginary eigenvalues, then there is a homeomorphism h defined on some neighborhood U of $\overline{x}$ in $\mathbb{R}^n$ locally taking orbits of the nonlinear flow Φ_t of $\dot{x} = f(x)$ to those of the linear flow $e^{tDf(\overline{x})}$ of $\dot{y} = Df(\overline{x})y$. The homeomorphism preserves the sense of orbits and can also be chosen to preserve parametrization by time.*

The index of a hyperbolic fixed point is the dimension of the stable manifold. In the context of our model, and given that we have two predetermined variables, equilibrium $\overline{x}$ is saddle-point stable if and only if $ind(\overline{x}) = 2$. The implications for our model appear in Tables A.1 and A.2 below, where rr_i denotes the real part of eigenvalue $r_{i \cdot \cdot i=1,..,4.}$

Table A.1: Index and equilibria in a four-dimensional vector space with two predetermined variables

Sign(rr_1)	Sign(rr_2)	Sign(rr_3)	Sign(rr_4)	det(A)	Trace(A)	Index	Equilibrium
$+$	$+$	$+$	$+$	> 0	> 0	0	no-equilibrium
$-$	$+$	$+$	$+$	< 0	$\gtreqless 0$	1	no-equilibrium
$-$	$-$	$+$	$+$	> 0	$\gtreqless 0$	2	unique
$-$	$-$	$-$	$+$	< 0	$\gtreqless 0$	3	multiple
$-$	$-$	$-$	$-$	> 0	< 0	4	multiple

Table A.2: Index and equilibria in a three-dimensional vector space with two predetermined variables

Sign(rr_1)	Sign(rr_2)	Sign(rr_3)	det($\widehat{A}$)	tr($\widehat{A}$)	Index	Equilibrium
$+$	$+$	$+$	> 0	> 0	0	no-equilibrium
$-$	$+$	$+$	< 0	$\gtreqless 0$	1	no-equilibrium
$-$	$-$	$+$	> 0	$\gtreqless 0$	2	unique
$-$	$-$	$-$	< 0	< 0	3	multiple

A linear approximation to eq. (21)-(24) near a hyperbolic fixed point

A linear approximation near the steady state reads

$$\dot{x}_t = B \times (x_t - x^*) \tag{36}$$

$$B \equiv \begin{bmatrix} 0 & -\frac{\sigma\alpha\widetilde{c}^*(\rho+\delta)}{\nu+R^*}f^* & -\frac{\sigma\widetilde{c}^*}{1+\frac{1}{\nu}R^*}f_k^{*2}\left(\beta+\frac{\tau^*}{1-\varphi^*}\right) & -\gamma\frac{\sigma(\rho+\delta)}{1-\tau^*}\frac{\widetilde{c}^*}{\widetilde{a}^*} \\ 0 & \rho+\delta+(\nu+R^*)\frac{\alpha-1}{\alpha} & \frac{\nu}{\alpha}\frac{f_k^{*2}}{f^*}\left(\beta+\frac{\tau^*}{1-\varphi^*}\right) & \gamma\frac{\nu}{\alpha}\frac{f_k^*}{f^*}\frac{1}{\widetilde{a}^*} \\ -1 & 0 & f_k^*-\delta & 0 \\ 0 & \alpha f^*\left[\frac{\alpha-1}{\alpha}\tilde{a}^*-\frac{1}{\nu}\right] & -f_k^*\left[\frac{1}{\nu}R^*+\beta+\tau^*\right] & \rho-\frac{\gamma}{\widetilde{a}^*} \end{bmatrix}$$

$$x_t \equiv \begin{bmatrix} c_t \\ \pi_t \\ k_t \\ a_t \end{bmatrix} \quad x^* \equiv \begin{bmatrix} c^* \\ \pi^* \\ k^* \\ a^* \end{bmatrix}.$$

(Asterisks denote steady-state levels; $f_k^*, f^*, \tilde{a}^*, \widetilde{c}^*$ are marginal product of capital, GDP, debt-to-GDP, and consumption-to-GDP, respectively; and $k_t, a_t,$ are predetermined state variables.)

We obtain analytically that the determinant of B is

$$-\left[\widetilde{c}^*\nu\sigma\rho f_k^{*2}\right]\frac{\alpha-1}{\alpha}\left[\beta+\frac{\tau^*}{1-\varphi^*}+\frac{\gamma}{\rho\tilde{a}^*}\left(\frac{1}{\nu}R^*-\frac{\tau^*\varphi^*}{1-\varphi^*}\right)\right],$$

and that the trace of B is

$$2\rho+(\nu+R^*)\frac{\alpha-1}{\alpha} + (\rho+\delta)\frac{1+\frac{1}{\nu}R^*}{1-\tau^*}-\frac{\gamma}{\tilde{a}^*}.$$

Proof of Proposition 5 Assume that $\phi_\alpha\left[\phi_\beta+\phi_\gamma\left(\mathcal{RSF}^*-1\right)\frac{\tau^*}{1-\varphi^*}\right] \neq 0.$

Then the multiple of eigenvalues is nonzero, which indicates that there is no zero eigenvalue. Assume now that $\phi_\alpha \left[\phi_\beta + \phi_\gamma \left(\mathcal{RSF}^{*}-1\right)\frac{\tau^*}{1-\varphi^*}\right] = 0$. Then either $\phi_\alpha = 0$ or $\left[\phi_\beta + \phi_\gamma \left(\mathcal{RSF}^{*}-1\right)\frac{\tau^*}{1-\varphi^*}\right] = 0$.

In what follows we show that either policy induces a zero eigenvalue, i.e. that there is a bifurcation at $\phi_\alpha = 0$ and given ϕ_β there is a bifurcation at $\phi_\gamma = \frac{\phi_\beta}{-\left(\mathcal{RSF}^{*}-1\right)\frac{\tau^*}{1-\varphi^*}}$. Note that simultaneously implementing $\phi_\alpha = 0$ and $\left[\phi_\beta + \phi_\gamma \left(\mathcal{RSF}^{*}-1\right)\frac{\tau^*}{1-\varphi^*}\right] = 0$ brings about a codimension two bifurcation.

Assume $\phi_\alpha = 0$ and $\left[\phi_\beta + \phi_\gamma \left(\mathcal{RSF}^{*}-1\right)\frac{\tau^*}{1-\varphi^*}\right] \neq 0$.

Substituting $\phi_\alpha = 0$ into equation (36) we obtain that

$$B_{[\phi_\alpha=0]} \equiv \begin{bmatrix} 0 & -\frac{\sigma\alpha\widetilde{c}^*(\rho+\delta)}{\nu+R^*}f^* & -\frac{\sigma\widetilde{c}^*}{1+\frac{1}{\nu}R^*}f_k^{*2}\left(\beta+\frac{\tau^*}{1-\varphi^*}\right) & -\gamma\frac{\sigma(\rho+\delta)}{1-\tau^*}\frac{\widetilde{c}^*}{\widetilde{a}^*} \\ 0 & \rho+\delta & \frac{\nu}{\alpha}\frac{f_k^{*2}}{f^*}\left(\beta+\frac{\tau^*}{1-\varphi^*}\right) & \gamma\frac{\nu}{\alpha}\frac{f_k^*}{f^*}\frac{1}{\widetilde{a}^*} \\ B_{3,1} & 0 & B_{3,3} & 0 \\ 0 & B_{4,2} & B_{4,3} & B_{4,4} \end{bmatrix}$$

where $B_{i,j}$ $i,j = 1,..4$ are components of B specified in eq. (36), respectively. Where $\phi_\alpha = 0$ the first row becomes a multiplication of the second row by $-\frac{\sigma\alpha\widetilde{c}^*}{\nu+R^*}f^*$. Consequently $B_{[\phi_\alpha=0]}$ is singular.

Assume $\phi_\alpha \neq 0$ and $\left[\phi_\beta + \phi_\gamma \left(\mathcal{RSF}^{*}-1\right)\frac{\tau^*}{1-\varphi^*}\right] = 0$.

Substituting $\widetilde{\gamma} \equiv \rho\tilde{a}^*\frac{\phi_\beta}{-\left(\mathcal{RSF}^{*}-1\right)\frac{\tau^*}{1-\varphi^*}}$ into equation (36) we obtain that

$$B_{[\widetilde{\gamma}]} \equiv \begin{bmatrix} 0 & B_{1,2} & B_{1,3} & \psi B_{1,3} \\ 0 & B_{2,2} & B_{2,3} & \psi B_{2,3} \\ B_{3,1} & 0 & B_{3,3} & 0 \\ 0 & B_{4,2} & B_{4,3} & \psi B_{4,3} \end{bmatrix}$$

where $\psi \equiv \frac{\rho}{-f_k^*(\mathcal{RSF}^*-1)\frac{\tau^*}{1-\varphi^*}}$ is a constant. It is straightforward to notice that the determinant of $B_{[\widetilde{\gamma}]}$ equals zero. Thus, a monetary-fiscal regime such that $\phi_\alpha\left[\phi_\beta+\phi_\gamma\left(\mathcal{RSF}^*-1\right)\frac{\tau^*}{1-\varphi^*}\right]=0$ brings about a non hyperbolic equilibrium. This concludes the first part of the proof.

The second part is straightforward. From Table A.1 it follows that a necessary condition for equilibrium determinacy in a 4x4 system is $\det(B)>0$. The proof of the proposition is concluded by requiring that the right-hand side of equation (30) is positive.

Proof of Lemma1 By definition all non-Ricardian fiscal stances exhibit $\phi_\gamma<1$. Since the rate of self financing of tax cuts does not exceed one we obtain that $\phi_\gamma\left(\mathcal{RSF}^*-1\right)>-1$ and as a result non-Ricardian regimes satisfy $\phi_\beta+\phi_\gamma\left(\mathcal{RSF}^*-1\right)\frac{\tau^*}{1-\varphi^*}>0 \Leftrightarrow \phi_\beta-\frac{\tau^*}{1-\varphi^*}>0\Leftrightarrow \beta>0$. As we assumed that β is always positive [see eq. (13)] we conclude the proof. *QED*.

Proof of Proposition 6 Equilibrium is determinate only where $ind(\overline{x})=2$. $B_{[\gamma=0]}$ is block recursive with one positive eigenvalue at the lower right 1×1 submatrix, and so we obtain the dimension of the stable manifold only by examining $\widehat{B_1}$. Observe Table A.2. $\det(\widehat{B_1})>0$ is a necessary condition. Furthermore, we must rule out the case where $ind(\overline{x})=0$ by requiring that policy also induce $\mathrm{tr}(\widehat{B_1})<0$. To conclude, we can ensure that $ind(\overline{x})=2$ by implementing a policy that brings about $r_1 r_2 r_3 > 0$ and $r_1 + r_2 + r_3 < 0$.

It follows from equation (34) that $r_1 r_2 r_3 > 0 \Leftrightarrow \phi_\alpha \phi_\beta < 0$. So $\det(\widehat{B_1}) > 0$ under two regimes: $\{\phi_\alpha > 0$ and $\phi_\beta < 0\}$ or $\{\phi_\alpha < 0$ and $\phi_\beta > 0\}$. We rule out the first regime, first, because $\phi_\beta < 0$, and second, because we obtain from equation (35) that in this regime $\mathrm{tr}(\widehat{B_1}) > 0$, and we cannot rule out the possibility of No-equilibrium. By contrast, we can ensure that $\mathrm{tr}(\widehat{B_1}) < 0$ and that $\det(\widehat{B_1}) > 0$ by requiring that $\phi_\alpha < -\frac{f_k^* + \rho}{\nu + R^*}$. *QED.*

Proof of Lemma2

Preliminaries

Theorem 1 [Hirsch and Smale (1976) Chap.16] *Let $f : W \to E$ be a C^1 vector field and $\overline{x} \in W$ an equilibrium of $\dot{x} = f(x)$ such that $Df(\overline{x}) \in L(E)$ is invertible. Then there exists a neighborhood $U \subset W$ of $\overline{x}$ and a neighborhood $\Re \subset \mho(W)$ of f such that for any $g \in \Re$ there is a unique equilibrium $\overline{y} \in U$ of $\dot{y} = g(y)$. Moreover, if E is normed, for any $\epsilon > 0$ we can choose $\Re$ so that $|\overline{y} - \overline{x}| < \epsilon$.*

Theorem 2 [Hirsch and Smale (1976) Chap.16] *Suppose that $\overline{x}$ is a hyperbolic equilibrium. In Theorem 1, then, $\Re$ and U can be chosen so that if $g \in \Re$, the unique equilibrium $\overline{y} \in U$ of $\dot{y} = g(y)$ is hyperbolic and has the same index as $\overline{x}$.*

Proof Consider now complex fiscal rules that exhibit $\gamma \neq 0$. In what follows I show that for small perturbations of γ near $\gamma = 0$ the system is structurally stable. Consider the system $\dot{x}_t = g_{[\gamma]}(x_t)$ where $\gamma = 0 + \varepsilon$,

$\varepsilon > 0$. Then a linearization reads

$$\dot{x}_t = \left[B_{[\gamma=0]} + \varepsilon\Delta\right] \times (x_t - x^*) \tag{37}$$

where

$$\Delta \equiv \begin{bmatrix} 0 & 0 & 0 & -\frac{\sigma(\rho+\delta)}{1-\tau^*}\frac{\widetilde{c}^*}{\widetilde{a}^*} \\ 0 & 0 & 0 & \frac{\nu}{\alpha}\frac{f_k^*}{f^*}\frac{1}{\widetilde{a}^*} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\widetilde{a}^*} \end{bmatrix}.$$

Since $B_{[\gamma=0]}$ is invertible, by implicit function theorem, $\dot{x}_t = g_{[\gamma]}(x_t)$ continues to have a unique solution $x^{**} = x^* + O(\varepsilon)$ near x^* for sufficiently small ε. Moreover, since we restrict attention to a set of policies that satisfy proposition 4, we ensure that $B_{[\gamma=0]} + \varepsilon\Delta$ is invertible, which implies that $x^{**} = x^*$ is the unique solution to equation (37). Furthermore, since the matrix of the linearized system $B_{[\gamma=0]} + \varepsilon\Delta$ has eigenvalues that depend continuously on ε, no eigenvalues can cross the imaginary axis if ε remains small with respect to the magnitude of the real parts of the eigenvalues of $B_{[\gamma=0]}$. Thus, the perturbed system (37) has a unique fixed point with eigenspaces and invariant manifolds of the same dimensions as those of the unperturbed system, and with an ε that is close in position and slope to the unperturbed manifolds. The main idea of this proposition is that perturbations are in the parameter space $\{\gamma\}$. By construction such perturbations do not change the steady state itself. However, they may change the phase portrait of the steady state. Thus, starting from a determinate equilibrium, as long as γ

does not reach its bifurcation point, the phase portrait of the (unchanged) steady state should not be affected by the perturbation. The formal proof of Corollary 4 follows directly from the following Theorems. Specifically, in the terminology of Theorems 1 and 2, I choose $\overline{y} = \overline{x}$ and $g(y)$ that differs from $f(x)$ up to the perturbation of γ.

Proof of Proposition 7 Consider a regime $\{(\phi_{\overline{\alpha}}, \phi_{\overline{\beta}}, \phi_{\overline{\gamma}}) \mid \phi_{\overline{\alpha}} < 0, \phi_{\overline{\beta}} > 0, \phi_{\overline{\gamma}} = 0\}$ that brings about a unique bounded rational expectations equilibrium near a hyperbolic steady state x^*. Then, perturbations to ϕ_γ in the neighborhood of $\phi_{\overline{\gamma}} = 0$ do not change the phase portrait of x^* as long as ϕ_γ is not perturbed until its bifurcation point. Bifurcations occur when ϕ_γ satisfies $\phi_\beta + \phi_\gamma \left(\mathcal{RSF}^* - 1\right) \frac{\tau^*}{1-\varphi^*} = 0$ which implies the bifurcation value $\overline{\phi_\gamma} = \frac{\phi_\beta}{(1-\mathcal{RSF}^*)\frac{\tau^*}{1-\varphi^*}}$.
Substituting in the expressions for ϕ_β and $\mathcal{RSF}^*$ [see eq. (29)]) yields that $\overline{\phi_\gamma} = \frac{\beta + \frac{\tau^*}{1-\varphi^*}}{-\frac{1}{\nu} R^* + \frac{\tau^* \varphi^*}{1-\varphi^*}}$.
Note that $\varphi^* < 1$ and $\beta, R^* > 0$. Thus the numerator at the right hand side of the last inequality is greater than the denominator and the denominator is positive. As a result, $\overline{\phi_\gamma} > 1$. Since non-Ricardian regimes exhibit $\phi_\gamma < 1 < \overline{\phi_\gamma}$ they are within the range of perturbations that do not change the phase portrait of x^*. QED.