Solving Boundary Value Problems in the Fiscal Theory of the Price Level

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Abstract

This paper specifies determinacy regions in the parameter space of monetary and fiscal policy interactions in economies with finance and tax distortions. It shows that the initial valuation of government debt is the fixed point of a continuous mapping that takes a closed interval on the real line into itself. It implements the Krasnoselski-Mann-Baily theorem to compute the equilibrium real value of nominal government debt. This computation indicates whether policy interactions are sustainable or lead to a default. The model exhibits nominal determinacy if and only if it exhibits real determinacy. Distorting taxes have dramatic effect on determinacy regions. Admissible monetary-fiscal policy interactions vary as the economy approaches the peak of its Laffer curve: the range of active fiscal responses to government debt narrows, whereas passive fiscal stances become inconsistent with equilibrium. Furthermore, policy targets vary when regimes switch from passive fiscal stances to active fiscal stances. Whereas passive fiscal stances focus entirely on secondary deficits, active fiscal stances should focus mainly on primary deficits.

JEL Codes: C62; C68; E42; E62; E63; H60.

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1 Introduction

In conventional macroeconomics, where the price level is determined by balance between money supply and money demand, a unique solution is obtained only where there are as many linearly independent boundary conditions as there are linearly independent predetermined variables [see Blanchard and Kahn (1980), Buiter (1984)]. This restriction runs the risk of drawing wrong conclusions when there are missing initial conditions. By contrast, in the fiscal theory of prices whose canonical foundations are set in Sargent and Wallace (1981), Leeper (1991), Sims (1994), Woodford (1995), and Bassetto (2002), the initial value of government liabilities is not given. Instead, a boundary condition exhibits how should the real value of government liabilities evolve along an equilibrium trajectory. Sims (2013) emphasizes that our thinking about inflation, monetary policy, and fiscal policy should be based on models of the later type. This paper presents a solution method for variants of the Blanchard and Kahn's (1980) method with a significant extension that is typical to the fiscal theory of prices. The contribution of this paper is to solve this set of problems as boundary value problems rather than the common practice in conventional macroeconomics which emphasizes initial value problems.

In the spirit of Sims (2013), we show how to specify fiscal-monetary policy interactions that induce an equilibrium where a unique valuation of financial wealth satisfies the boundary condition. We then show how to calculate its unique initial value. It is widely known that Schauder's fixed-point theorem is a powerful method for proving existence theorems. If one wishes to prove that a given problem has a solution, he proceeds by associating it with a convex compact set E in some Banach space, and a continuous transformation T which carries E into itself. Schauder's theorem asserts that T must have at least one fixed-point in E. Developments attributed to Krasnoselski (1955), Mann (1953), and Bailey (1974) show that if E is a closed interval on the real line and T is continuous than the fixed point is unique in E and can be obtained as a limit to a sequence of points, whereas the limit is obtained via an iteration process. We implement this method on the boundary to obtain the equilibrium valuation of government debt in a fiscal theoretic framework. We assume a production economy with finance constraints and distorting taxes. These assumptions add complications to the conventional model but have the merit of clarifying a few issues that the fiscal theory leaves obscure. The distortions break linear dependence between the boundary conditions. In the fiscal theoretic framework the real value of government debt equals the present value of the sum of future surpluses. However, to compute this stream of surpluses we must know the initial value of non capital wealth. Thus, computing the entire equilibrium involves a simultaneous determination of the initial valuation of financial wealth and the entire state space. This restriction indicates that in equilibrium the initial value of financial wealth is a solution to a fixed point problem. We then prove that the initial value of financial wealth is a fixed point of a continuous mapping that takes a closed interval of the real line into itself and, as a result, is a limit to a Krasnoselski-Mann-Bailey sequence.

The main results of this paper are: The issues of nominal and real determinacy are inseparable. The model economy exhibits nominal determinacy if and only if it exhibits real determinacy. When taxes are lump-sum and decisions are distorted only by finance constraints, the type of monetary-fiscal policy interactions that stabilize the economy are the usual active monetary-passive fiscal/ passive-monetary active-fiscal decoupling. By contrast, tax distortions have a dramatic effect on determinacy regions. The result that an active monetary policy must interact with a passive fiscal policy and that a passive monetary policy must interact with an active fiscal policy carries to the economy with tax distortions. However, parameters ranges change and admissible interactions become complex. Firstly, the cutoff response to government debt below which fiscal policy is considered active depends on debt-to-GDP target. Secondly, tax rate responses to output matter. Finally, a Laffer curve emerges endogenously and admissible policy interactions depend on whether the economy resides near or away from the peak of its Laffer curve. Admissible monetary-fiscal policy interactions vary as the economy approaches the peak of its Laffer curve. The range of active fiscal responses to government debt narrows near the peak of the Laffer curve whereas passive fiscal stances become inconsistent with equilibrium.

The rest of the paper is organized as follows. Section 2 describes the economic environment, the optimal decision-making of the representative household and the evolution of government debt. In this section we introduce liquidity constraints on all transactions but adhere to the assumption that the government has access to lump-sum taxation. Section 3 describes the aggregate dynamics of the economy and obtains necessary and sufficient conditions for a unique determination of the price level and the entire equilibrium trajectory. Results show that regions of equilibrium determinacy in this variant are identical to the canonical result of Leeper (1991). In section 4 we assume that the government has access only to a distorting taxation technology and describe the dynamic general equilibrium of the economy. In section 5 we obtain equilibrium determinacy regions in the policy parameters space where fiscal policy is active. We also specify the algorithm for obtaining an equilibrium fiscal theoretic initial valuation of nominal government liabilities. In section 6 we obtain equilibrium determinacy regions where fiscal policy is passive. All proofs are deferred to the appendix.

2 A Model with Lump-Sum Taxes and Finance Constraints

Time is continuous. The economy is closed and populated by a continuum of identical infinitely long-lived households, with measure one. The representative household enjoys consumption, is endowed with perfect foresight and one unit of time per "period". The representative household inelastically supplies it's endowment of labor, so it's lifetime utility is given by

$$U_t = \int_t^{\infty} e^{-\rho s} u(c_s) ds \tag{1}$$

where $\rho > 0$ denotes the rate of time preference, c_s denotes consumption per capita, $u(\cdot)$ is twice differentiable, strictly increasing, strictly concave, and satisfies the usual limit conditions. Production takes place in a competitive sector via a constant returns to scale production technology $f(k_t)$ where k_t denotes per capita capital which depreciates at a rate δ. Finally, $f(k_t)$ is concave and twice differentiable. Money enters the economy via a liquidity constraint on all transactions. Let m_t denote the stock of money denominated in the consumption good, then a formal representation of the liquidity constraint is:

$$c_t + I_t \leq \nu m_t \tag{2}$$

where I_t denotes per-capita investment and ν is money velocity.[1] Assuming the existence of nominal government bonds, the representative household's budget constraint is

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[1] Let $\frac{1}{\nu}$ denote the inverse of money velocity, then a requirement that $\int_t^{t+\frac{1}{\nu}} [c(s) + I(s)]\, ds \leq m_t$ formalizes the liquidity constraint. A Taylor series expansion gives $\int_t^{t+\frac{1}{\nu}} [c(s) + I(s)]ds = \frac{1}{\nu}[c(t) + I(t)] + \frac{1}{2}\left(\frac{1}{\nu}\right)^2 [\dot{c}(t) + \dot{I}(t)] + \cdots$ and $\frac{1}{\nu}(c + I) \leq m$ can be interpreted as a first-order approximation.

$$c_t + I_t + \dot{b}_t + \dot{m}_t = (R_t - \pi_t)b_t - \pi_t m_t + f(k_t) + T_t - \tau_t^L \tag{3}$$

where b_t is a real measure of the stock of interest bearing government bonds, R_t is the nominal rate of interest, π_t is the rate of inflation, T_t, τ_t^L are real lump sum transfers and taxes, respectively. Altogether, the household's lifetime maximization problem becomes

$$V[b_t, m_t, k_t] = Max_{\{c_s, I_s, x_s\}_{s=t}^{\infty}} \int_t^{\infty} e^{-\rho s} u(c_s) ds \tag{4}$$

$$s.t.$$

$$\dot{b}_s = (R_s - \pi_s)b_s - \pi_s m_s + f(k_s) + T_s - I_s - c_s - x_s - \tau_s^L$$

$$\dot{m}_s = x_s$$

$$\dot{k}_s = I_s - \delta k_s$$

$$c_s + I_s \leq \nu m_s$$

$$a_s, k_s \geq 0$$

With a borrowing constraint such that $lim_{t \to \infty} a_t e^{-\int_0^t [R_s - \pi_s] ds} \geq 0$ where $a_t \equiv b_t + m_t$ denotes the representative household's non capital wealth.

2.1 The optimal program

Each household chooses sequences of $\{c_t, I_t, x_t\}$ so as to maximize its lifetime utility, taking as given the initial stock of capital k_0, the initial stock of financial wealth a_0, and the time path $\left\{\tau_t^L, T_t, R_t, \pi_t\right\}_{t=0}^{\infty}$ which is exogenous from the view point of a household. The necessary conditions for an interior maximum are

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\begin{aligned}
u'(c_t) &= \lambda_t(1 + \frac{1}{\nu} R_t) && \text{(5a)} \\
\mu_t &= u'(c_t) && \text{(5b)} \\
\zeta_t &= \frac{1}{\nu} R_t \lambda_t && \text{(5c)} \\
\zeta_t(\nu m_t - c_t - I_t) &= 0; \zeta_t \geq 0 && \text{(5d)}
\end{aligned}
$$

Where λ_t, μ_t are time-dependent co-state variables interpreted as the marginal valuations of financial wealth and capital, respectively; ζ_t is a time-dependent Lagrange multiplier associated with the liquidity constraint and equation (5d) is a Kuhn-Tucker condition.

Following Benhabib et. al. (2001b,2002), I restrict attention to a positive nominal interest rate. As a result, equation (5c) implies that ζ_t is positive.

It then follows from (5d) that $m_t = \frac{1}{\nu}(c_t + I_t)$ which implies that where the nominal interest rate is positive the liquidity constraint is binding. Second, and after substituting $m_t = \frac{1}{\nu}(c_t + I_t)$ and $a_t = b_t + m_t$ into equation (3), the state and co-state variables must evolve according to

$$\dot{\lambda}_t = \lambda_t [\rho + \pi_t - R_t] \tag{6}$$

$$\dot{\mu}_t = -\lambda_t f'(k_t) + (\rho + \delta)\mu_t \tag{7}$$

$$\dot{k}_t = I_t - \delta k_t \tag{8}$$

$$\dot{a}_t = (R_t - \pi_t)a_t + f(k_t) + T_t - \tau_t^L - (c_t + I_t)\left(1 + \frac{1}{\nu}R_t\right) \tag{9}$$

Solving equation (9) yields that the household's intertemporal budget constraint is of the form

$$lim_{t\to\infty} e^{-\int_0^t [R_s - \pi_s]ds} a_t = a_{0+} \int_0^{\infty} e^{-\int_0^t [R_s - \pi_s]ds} \left[f(k_t) + T_t - \tau_t^L - (c_t + I_t)\left(1 + \frac{1}{\nu}R_t\right)\right] dt \geq 0 \tag{10}$$

and the condition that his intertemporal budget constraint holds with equality yields the usual transversality condition:

$$lim_{t\to\infty} a_t e^{-\int_0^t [R_s - \pi_s] ds} = 0 \tag{11}$$

Equations (6) – (11) fully describe the optimal program of a representative household for which the time path $\left\{\tau_t^L, T_t, R_t, \pi_t\right\}_{t=0}^{\infty}$ is exogenously given.

2.2 The government and the evolution of government debt

The government consists of a fiscal authority and a monetary authority. The consolidated government prints money, M_t, issues nominal bonds, B_t, collects lump-sum taxes to the amount τ_t^L, and rebates to the households a lump-sum transfer T_t. Its dollar denominated budget constraint is therefore given by $R_t B_t + P_t T_t = \dot{M}_t + \dot{B}_t + P_t \tau_t^L$ where P_t is the nominal price of a consumption bundle. It is assumed that the monetary authority imposes a desired interest rate, R_t, and that the fiscal authority can continuously control lump-sum taxes and transfers.

Dividing both sides of the nominal budget constraint by P_t and rearranging, yield that government liabilities, denoted by $a_t \equiv b_t + m_t$, evolve according to:

$$\dot{a}_t = (R_t - \pi_t)\, a_t - R_t m_t + \left(T_t - \tau_t^L\right) \tag{12}$$

where $\pi_t \equiv \frac{\widehat{\dot{P}_t}}{P_t}$ and the hatted time derivative is a right derivative, referring to expected inflation from now on. Equation (12) captures the following story: The consolidated budget is not necessarily balanced, and deficits (surpluses) are financed via increments (decrements) to government debt. As a result, government liabilities increase with the primary deficit, $\left(T_t - \tau_t^L\right)$, and with the real interest paid over outstanding debt, $(R_t - \pi_t)\, a_t$, and decrease with seigniorage, $R_t m_t$.

2.2.1 Monetary Policy

Following Bassetto (2002) and in order to avoid criticism in the spirit of Buiter (2002) that may arise when money supply rules interact with non-Ricardian fiscal rules, we assume that monetary policy is implemented via an interest rate rule. Specifically, monetary policy implements a simple version of a Taylor rule that emphasizes inflation targeting,

$$R(\pi_t) = \rho + \pi^* + \alpha(\pi_t - \pi^*) \tag{13}$$

where $\alpha > 0$ is a constant that measures the responsiveness to increases in inflation around π^*. Following Leeper (1991) and Benhabib et. al. (2001a, 2001b) this rule underlies most of the recent monetary policy literature. When $\alpha > 1$ monetary policy is considered hawkish, responding to deviations of the expected inflation from the inflation target by sharply raising the nominal interest rate with the aim of stabilizing inflation around π^*. In Leeper's (1991) terminology this is called an "active" monetary policy whereas a weak response such that $\alpha < 1$ is called "passive" monetary policy.

2.2.2 Fiscal Policy

Fiscal policy emphasizes output targeting and debt targeting,

$$\tau^L(k_t, a_t) = \tau^{L*} + \beta\left[f(k_t) - f(k^*)\right] + \gamma\left[a_t - a^*\right] \tag{14}$$

$$\forall t \ : \ T_t = \overline{T} \tag{15}$$

The correlation between the lump-sum tax and the secondary deficit is captured by γ. In the terminology of Leeper (1991) fiscal policy is considered 'active ' if it lets government liabilities grow at a rate greater than the real interest rate, i.e. $\gamma < \rho$. Timing is also important. All the debt is inherited from the past and the government is only setting the terms to repay it. Note that fiscal policy responds only after real output and the real value of government liabilities had been realized, whereas monetary policy responds to expected changes in nominal prices. Accordingly, we consider two stocks - the capital stock, and the real value of government liabilities - as predetermined variables.

3 General Equilibrium

In equilibrium, the goods market clears

$$f(k_t) = c_t + I_t \tag{16}$$

assuming a positive nominal interest rate the money market clears such that

$$m_t = \frac{1}{\nu}(c_t + I_t) \tag{17}$$

and government liabilities equal household's assets.

Using the monetary policy rule, imposing market clearing conditions, and assuming that the elasticity of intertemporal substitution in consumption is constant, we arrive at the following characterization of the general equilibrium of the economy:

Proposition 1 *In equilibrium, the aggregate dynamics satisfy the following ODE system:*

$$\begin{aligned}
\frac{\dot{c}_t}{c_t} &= \sigma\left[\frac{1}{1+\frac{1}{\nu}R\left(\pi_t\right)}f'(k_t)-(\rho+\delta)\right] && (18)\\
\dot{\pi}_t &= \frac{\nu+R\left(\pi_t\right)}{\alpha}\left\{\left[R\left(\pi_t\right)-\pi_t\right]-\left[\frac{1}{1+\frac{1}{\nu}R\left(\pi_t\right)}f'(k_t)-\delta\right]\right\} && (19)\\
\dot{k}_t &= f(k_t)-c_t-\delta k_t && (20)\\
\dot{a}_t &= \left[R\left(\pi_t\right)-\pi_t\right]a_t-\frac{1}{\nu}R\left(\pi_t\right)f(k_t)+\left[\overline{T}-\tau^L(k_t,a_t)\right] && (21)
\end{aligned}$$

Equation (18) is an Euler equation, where $\sigma > 0$ denotes the elasticity of intertemporal substitution in private consumption. In our economy the marginal product of capital is distorted by the liquidity constraint. Notice that with no distortion equation (18) is reduced to the familiar Ramsey type Euler equation. Equation (19) was obtained by taking a time derivative from the first order condition ($5a$) and substituting in equation (6). It corresponds to a Fisher equation in which the nominal rate of interest varies with expected inflation and the real rate of interest. Note that the effect of liquidity constraints on investment are similar to those of adjustment costs. As a consequence, an instantaneous no-arbitrage condition between capital assets and financial assets does not hold. However, according to equation (19), in equilibrium an intertemporal no-arbitrage prevails. Finally, equations (20)-(21) were obtained by substituting market clearing conditions (16)-(17) into equations (8)-(9). At this point, the set of equations (18)-(21) internalizes the government's policies and market clearing and can therefore be viewed as the solution to the central planner's problem.

Definition *A perfect-foresight equilibrium with lump-sum taxes and finance constraints is a set of sequences* $\left\{c_t, \pi_t, k_t, a_t, \tau_t^L, T_t, R_t\right\}$ *and an initial price level* $P_0 > 0$ *satisfying (16)-(21) given* $M_0 + B_0 > 0$ *and* $k_0 > 0$.

3.1 Steady-State Equilibrium

It follows from equation (18) that in a steady state

$$f'(k^*) = (\rho + \delta)\left(1 + \frac{1}{\nu}R^*\right) \tag{22}$$

where R^* is a steady state rate of interest. From equations (19) and (22), R^* must satisfy

$$R^* = \rho + \pi^* \tag{23}$$

where π^* is the long-run rate of inflation. Equation (20) implies that the steady state consumption is

$$c^* = f(k^*) - \delta k^* \tag{24}$$

Finally, it follows from equation (21), that in a steady-state equilibrium government liabilities must satisfy $a^* = \frac{1}{\rho}\left[f(k^*)\frac{1}{\nu}R^* + \tau^{L*} - \overline{T}\right]$. Let $\widetilde{a}^* \equiv \frac{a^*}{f(k^*)}$, $\widetilde{s}^* \equiv \frac{\tau^{L*} - \overline{T}}{f(k^*)}$ denote liabilities to GDP and surplus to GDP in the steady state, respectively, then we obtain that a sustainable debt level must satisfy

$$\widetilde{a}^* = \frac{1}{\rho}\left[\frac{1}{\nu}R^* + \widetilde{s}^*\right] \tag{25}$$

Note that an equilibrium trajectory $\{c_t, \pi_t, k_t, a_t, \tau_t, T_t, R_t\}_0^\infty$ should converge to the steady state $\{c^*, \pi^*, k^*, a^*, \tau^*, T^*, R^*\}$ whereas the aggregate dynamics in equations (18) - (21) imply that a steady-state equilibrium determines only 4 variables.[2] It is thus straightforward to show the following:

[2] Equations (22)-(25) demonstrate that we have four equations and seven variables.

Proposition 2 *A necessary condition for steady state determination is that the government proclaims three explicit targets.*

Proposition 2 follows mainly from equation (25). The steady state is sustained only if the revenues from taxes and seigniorage equal the sum of transfers and debt service. Thus, as equation (25) links $\pi^*, \tau^{L*}, a^*, \overline{T}$ to a balanced budget condition, three targets should be specified explicitly, and the fourth is implied by the stipulation to run a balanced budget in the steady state. Proposition 2 has the following implications:

Corollary 1 *Where government proclaims only fiscal targets such as* $\left\{\widetilde{a}^*, \widetilde{s}^*, \overline{T}\ \right\}$, *the inflation target is implied according to* $\pi^* = \nu\left(\rho\widetilde{a}^* - \widetilde{s}^*\right) - \rho$ *and lump-sum taxes are then set according to* $\tau^{L*} = \overline{T} + s^*$

Corollary 2 *Where government proclaims that the explicit set of targets is* $\left\{\tau^{L*}, \pi^*, \overline{T}\right\}$, $\widetilde{a}^*$ *is implied directly by condition (25).*

In the rest of the section I assume that the government explicitly proclaims $\left\{\tau^{L*}, \pi^*, \overline{T}\right\}$.

3.2 Equilibrium Dynamics

3.2.1 Price level determination

Solving equation (21), and letting $t \to \infty$ yields the well known assertion that market equilibrium requires intertemporal government budget balance:

Lemma 1 *In equilibrium*

$$0 = \lim_{t\to\infty} e^{-\int_0^t [R(\pi_s)-\pi_s]ds} a_t = a_0 - \int_0^\infty e^{-\int_0^t [R(\pi_s)-\pi_s]ds} \left\{ \tfrac{1}{\nu} R(\pi_t) f(k_t) + \tau^L(k_t, a_t) - \overline{T} \right\} dt$$

Lemma 1 follows from: a) solving equation (21) which internalizes the idea that in equilibrium households' assets equal government's liabilities; and b) imposing conditions (10)-(11) that the households' intertemporal budget constraint holds with equality. Note that substituting the fiscal rule (14) into (21) yields that government liabilities evolve according to:

$$\dot{a}_t = [R(\pi_t) - \pi_t - \gamma] a_t - f(k_t) \left[\frac{1}{\nu} R(\pi_t) + \beta \right] + \left[\overline{T} - \tau^{L*} + \beta f(k^*) + \gamma a^* \right] \tag{26}$$

Solving equation (26) for a_t we obtain that:

$$Q_t a_t = a_0 - \int_0^t Q_s \left\{ f(k_s) \left[\frac{1}{\nu} R(\pi_s) + \beta \right] + S^{**} \right\} ds \tag{27}$$

where $Q_t \equiv e^{-\int_0^t [R(\pi_s) - \pi_s - \gamma] ds}$ is a discount factor and $S^{**} \equiv \tau^{L*} - \left[\overline{T} + \beta f(k^*) + \gamma a^* \right]$ sums all the constant terms in eq. (26). Letting $t \to \infty$ and rearranging we obtain that:

$$\begin{aligned} & \lim_{t\to\infty} e^{-\int_0^t [R(\pi_s) - \pi_s] ds} a_t \\ = & \lim_{t\to\infty} e^{-\gamma t} \left[a_0 - \int_0^t Q_s \left\{ f(k_s) \left[\frac{1}{\nu} R(\pi_s) + \beta \right] + S^{**} \right\} ds \right] \end{aligned} \tag{28}$$

Equation (28) has several implication. First, according to Lemma1 the left hand side of equation (28) equals zero in equilibrium. Thus, we must choose γ in a range so as to ensure that the right hand side of equation (28) equals zero. Second, above a certain value of γ, the discount factor, Q_s, is not contracting and as a result, the integral on the right hand side is not defined.

Proposition 3 *If* $\gamma < \rho$ *and equilibrium trajectory is unique the equilibrium price level is determined to satisfy*

$$\frac{B_0+M_0}{P_0} = \int_0^\infty Q_s \left\{ f(k_s) \left[\frac{1}{\nu} R(\pi_s) + \beta \right] + S^{**} \right\} ds$$

Proposition 3 shows how nominal prices are determined if the fiscal authority lets it's liabilities grow at a rate greater than the real interest rate. Essentially, if the government operates a fiscal rule such that the present discounted value of real government liabilities is not expected to vanish, the price level must play an active role in bringing about fiscal solvency in equilibrium. This idea has been emphasized by the fiscal theory of prices and is discussed extensively in Woodford (1995) and Cochrane (2001, 2005).

when the government operates a fiscal rule with $\gamma > \rho$ the level of nominal prices cannot be determined according to Proposition 3. The idea is that $\gamma > \rho$ implies that the government ensures that its liabilities will converge back to the target and for any price level fiscal solvency is ensured by the fiscal policy. In this case fiscal policy is considered Ricardian and the level of nominal prices is determined so as to clear the money market. Specifically, where $\gamma > \rho$, P_0 is determined according to equation (17) i.e. $\frac{M_0}{P_0} = \frac{1}{\nu} (c_0 + I_0)$.

3.2.2 Transitional Dynamics

In this section we characterize the monetary-fiscal interactions that induce a unique trajectory. According to equations (18)-(21) and the policy rules (13) - (15), all the variables are a mapping in the (c, π, k, a) space. A linear approximation to equations (18)-(21) near the steady state is obtained through the system

$$\dot{x_t} = A \times (x_t - \overline{x}) \tag{29}$$

where[3]

$$x_t \equiv \begin{bmatrix} c_t \\ \pi_t \\ k_t \\ a_t \end{bmatrix} \quad \overline{x} \equiv \begin{bmatrix} c^* \\ \pi^* \\ k^* \\ a^* \end{bmatrix} \quad A \equiv \begin{bmatrix} 0 & -\frac{\sigma\alpha\widetilde{c}^*(\rho+\delta)}{\nu+R^*} f^* & -\frac{\sigma\widetilde{c}^*}{1+\frac{1}{\nu}R^*}\frac{f_k^{*2}}{\xi_k} & 0 \\ 0 & \rho+\delta+(\nu+R^*)\frac{\alpha-1}{\alpha} & \frac{\nu}{\alpha}\frac{f_k^{*2}}{f^*\xi_k} & 0 \\ -1 & 0 & f_k^* - \delta & 0 \\ 0 & f^*\left[(\alpha-1)\tilde{a}^* - \frac{\alpha}{\nu}\right] & -f_k^*\left[\frac{1}{\nu}R^* + \beta\right] & \rho-\gamma \end{bmatrix}$$

π^* is a policy target proclaimed by the government denoting the long run level of inflation. c^*, k^*, a^* are steady state levels obtained by equations (22)-(25). $f_k^*, f^*, \tilde{a}^*, \widetilde{c}^*$ are marginal product of capital, GDP, debt to GDP, and

[3] Asterisk denote steady state levels.

consumption to GDP, respectively, and $\xi_k \equiv \frac{[f'(k^*)]^2}{f(k^*)f''(k^*)} < 0$ is the (constant) elasticity of production technology. When A has no eigenvalue with zero real part, the steady state $\overline{x} \equiv (c^*, \pi^*, k^*, a^*)'$ is a hyperbolic fixed point and according to Hartman-Grobman's Theorem and the Stable Manifold Theorem for a fixed point, the asymptotic behavior of solutions near it is determined by the linearization[4]. Let $r_i \quad i = 1, .., 4$ denote the eigenvalues of A, then by calculating the determinant and trace of A we obtain that:

$$r_1 r_2 r_3 r_4 = -\frac{\sigma \widetilde{c}^* (\nu + R^*)^2 (\rho + \delta)^2}{\alpha \nu \xi_k} (\alpha - 1)(\gamma - \rho) \tag{30}$$

$$r_1 + r_2 + r_3 + r_4 = -\gamma + 2\rho + (\nu + R^*)\frac{\alpha - 1}{\alpha} + (\rho + \delta)(1 + \frac{1}{\nu} R^*) \tag{31}$$

and Proposition 4 follows directly from equation (30),

Proposition 4 *The steady state $\overline{x}$ is hyperbolic if and only if*

$(\alpha - 1)(\gamma - \rho) \neq 0.$

Also note that the dynamic system (29) has two predetermined variables.

[4]See Guckenheimer and Holmes (1983) Theorems 1.3.1 and 1.3.2.

As a result, equilibrium is unique if, and only if, the system has two stable roots. Table A.1. in the appendix clarifies this issue. We can thus obtain a necessary condition for equilibrium determinacy:

Proposition 5 *A necessary condition for determinacy of equilibrium is*

$$(\alpha - 1)(\gamma - \rho) > 0.$$

Proposition 5 demonstrates that determinacy regions are decoupled. There are two regions of the parameter space that may deliver a unique equilibrium. Notice also that this result is independent of the parameter β and depends entirely on the monetary response towards inflation and on fiscal responses towards government liabilities. Specifically, Proposition 5 states that we can consider only the following determinacy regions: a monetary policy such that $\alpha > 1$ must interact with a fiscal policy such that $\gamma > \rho$, and a monetary policy such that $\alpha < 1$ must interact with a fiscal policy such that $\gamma < \rho$. In the former regime, henceforth Regime M, monetary policy emphasizes inflation targeting and fiscal policy stabilizes government debt whereas in the later regime, henceforth Regime F, fiscal policy resembles recent actual behavior in many countries. In regime M, deficit financed tax cuts or spending increases do not affect aggregate demand because households expect that the resulting increase in government liabilities will be matched by future tax increases. Thus, increases in government debt do not raise wealth. This fiscal behavior relieves monetary policy of fiscal financing leaving inflation targeting as the main concern of the central bank. In Regime F higher deficits do not create higher expected taxes and the central bank is expected to raise interest rates only weakly with inflation. Households therefore initially perceive the increase in government debt as an increase in their real wealth trying to convert higher wealth into consumption. Rising demand brings with it rising nominal prices and with a predetermined output nominal prices continue to rise until real wealth falls back to its pre deficit-rise level. By preventing interest rates from rising sharply with inflation monetary policy prevents debt service from growing too rapidly, which stabilizes the value of government debt.

With lump-sum taxation fiscal policy entails no distortions and is therefore transmitted to the economy purely through wealth effects. The fact that β has little effect on aggregate dynamics supports this claim. Note from equation (29) that β matters for aggregate dynamics only where the real value of government liabilities responds to output. Since the dynamics of (c, π, k) are independent of government liabilities, β has no effect on the region of determinacy. It comes to play in the speed at which nominal prices adjust under Regime F. This role is demonstrated by Proposition 3. I next discuss the sufficient conditions for equilibrium determinacy.

3.3 Stabilizing Monetary-Fiscal Interactions with Lump-Sum Taxes and Finance Constraints

Examining eq. (29), the dynamics of (c, π, k) are independent of government liabilities. This feature has two implications: (a) one eigenvalue of the (c, π, k, a) system is $\rho - \gamma$; (b) the remaining three eigenvalues are determined by $\widehat{A}$ the upper left 3×3 submatrix of A, so that the dynamics of (c, π, k) are completely determined by $\widehat{A}$. It is straightforward to show that the three remaining eigenvalues satisfy:

$$r_1 r_2 r_3 = \frac{\sigma \widetilde{c}^* (\nu + R^*)^2 (\rho + \delta)^2}{\alpha \nu \xi_k} (\alpha - 1) \tag{32}$$

$$r_1 + r_2 + r_3 = \rho + (\nu + R^*) \frac{\alpha - 1}{\alpha} + (\rho + \delta) (1 + \frac{1}{\nu} R^*) \tag{33}$$

And since the fourth eigenvalue equals $\rho - \gamma$ we are able to obtain Leeper's

(1991) result for this economy:

Proposition 6 *Two monetary -fiscal regimes induce a determinate equilibrium:*

I) Active-Monetary Passive -Fiscal where $\alpha > 1$ *and* $\gamma > \rho$. *In this regime nominal prices are pinned down so as to clear the money market.*

II) *Passive-Monetary Active -Fiscal where* $\alpha < \frac{1}{1+\frac{\rho}{\nu+R^*}+\frac{\rho+\delta}{\nu}}$ *and* $\gamma < \rho$. *In this regime nominal prices are pinned down according to Proposition 3.*

We have thus verified that Leeper's (1991) result obtains in a production economy with finance constraints and lump sum taxation.

4 A Model with Income Taxes and Finance Constraints

We now turn to the main part of the paper assuming that the government has no access to lump-sum taxation. This assumption adds a new channel for fiscal policy - an arbitrage channel. Adjustments to the tax rate change the after tax marginal product of capital and therefore may affect aggregate dynamics because of subsequent adjustments to the composition of wealth portfolio. With income taxation, the household's budget constraint becomes

$$c_t + I_t + \dot{b}_t + \dot{m}_t = (R_t - \pi_t)b_t - \pi_t m_t + (1 - \tau_t)f(k_t) + T_t \tag{34}$$

where $\tau_t \in [0, 1]$ is the income tax rate. The household's lifetime maximization problem in this economy becomes

$$\mathcal{J}\left[b_t, m_t, k_t\right] = Max_{\{c_s, I_s, x_s\}_{s=t}^{\infty}} \int_t^{\infty} e^{-\rho s} u(c_s) ds \tag{35}$$

$$s.t.$$

$$\dot{b}_s = (R_s - \pi_s)b_s - \pi_s m_s + (1 - \tau_s)f(k_s) + T_s - I_s - c_s - x_s$$

$$\dot{m}_s = x_s$$

$$\dot{k}_s = I_s - \delta k_s$$

$$c_s + I_s \leq \nu m_s$$

$$a_s, k_s \geq 0$$

With a borrowing constraint such that $lim_{t \to \infty} a_t e^{-\int\limits_0^t [R_s - \pi_s] ds} \geq 0$ where $a_t \equiv b_t + m_t$. As each household chooses sequences of $\{c_t, I_t, x_t\}$ so as to maximize lifetime utility, taking as given k_0, a_0 and the time path $\{\tau_t, T_t, R_t, \pi_t\}_{t=0}^{\infty}$ the necessary conditions for an interior maximum are are those described by equations (5a) - (5d). However, the income tax distorts intertemporal considerations and the state and co-state variables must now evolve according to

$$\dot{\lambda_t} = \lambda_t [\rho + \pi_t - R_t] \tag{36}$$

$$\dot{\mu_t} = -\lambda_t (1 - \tau_t) f'(k_t) + (\rho + \delta) \mu_t \tag{37}$$

$$\dot{k_t} = I_t - \delta k_t \tag{38}$$

$$\dot{a_t} = (R_t - \pi_t) a_t + (1 - \tau_t) f(k_t) + T_t - (c_t + I_t) \left(1 + \frac{1}{\nu} R_t\right) \tag{39}$$

Solving equation (39) yields that the household's intertemporal budget constraint is of the form

$$lim_{t\to\infty} e^{-\int_0^t [R_s-\pi_s]ds} a_t = a_{0+}\int_0^\infty e^{-\int_0^t [R_s-\pi_s]ds}\left[(1-\tau_t)f(k_t)+T_t-(c_t+I_t)\left(1+\frac{1}{\nu}R_t\right)\right]dt \geq 0 \tag{40}$$

and the condition that his intertemporal budget constraint holds with equality yields the transversality condition $lim_{t\to\infty} a_t e^{-\int_0^t [R_s-\pi_s]ds} = 0$. Equations (36) – (39) fully describe the optimal program of a representative household for which the time path $\{\tau_t, T_t, R_t, \pi_t\}_{t=0}^{\infty}$ is exogenously given.

4.1 The government and the evolution of government debt

The consolidated government prints money, M_t, issues nominal bonds, B_t, collects taxes to the amount of $\tau_t f(k_t)$ where τ_t denotes an income tax rate, and rebates to the households a lump-sum transfer T_t. Its dollar denominated budget constraint is therefore given by $R_t B_t + P_t T_t = \dot{M}_t + \dot{B}_t + P_t \tau_t f(k_t)$. Dividing both sides of the nominal budget constraint by P_t and rearranging, yields that government liabilities evolve according to:

$$\dot{a}_t = (R_t - \pi_t)\, a_t - R_t m_t + T_t - \tau_t f(k_t) \tag{41}$$

and we continue to assume that monetary policy is carried out according to eq. (13).

4.1.1 Fiscal Policy

Previous work with distorting taxes emphasizes that tax rates adjust to stabilize government debt. Prominent papers in this literature include Bi (2012), Bi and Traum (2012) and Bi, Leeper and Leith (2012). The later papers consider the non distortionary fiscal transfers as a residual in the government budget constraint, exogenously determined by an AR(1) process, whereas Bi (2012) assumes that lump-sum transfers are countercyclical. This assumption follows from the estimation that the elasticity of real detrended transfers with respect to detrended productivity is negative for all of the OECD countries. These modeling choices boil down to the idea that fiscal policy is responsive to deficits. Responses to variations in the deficit are then carried out by adjusting taxes or by adjusting transfers according to time invariant rules. Given that the state space is $\{k_t, a_t\}$, we assume an income tax rate rule of the form:

$$\tau(k_t, a_t) = \tau^* + \beta \frac{f(k_t) - f(k^*)}{f(k^*)} + \gamma \frac{a_t - a^*}{a^*} \tag{42}$$

and we continue to assume an active transfer policy such that $\forall t : T_t = \overline{T}$. Equation (42) exhibits the correlation of the income tax rate with the state variables. The correlation between the tax rate and the primary deficit is captured by β. The correlation between the tax rate and the secondary deficit is captured by γ. In our model, the threshold for γ under which fiscal policy should be considered as 'active ' is addressed in detail in following sections.

Limitations to the fiscal rule

Computing Equilibrium In general, allowing a tax policy to depend on endogenous variables allows us to examine interactions between tax rates and the economy's equilibrium. Income, consumption, and public debt clearly depend on the level of tax rates. But in our economy tax rates are chosen to depend on these variables. As a result, computing an equilibrium should involve a simultaneous determination of output, debt and tax rates. Coleman (1991) develops a methodology for examining discrete time models that can address these type of issues. Our model offers a different approach. In our model, computing an equilibrium involves a simultaneous determination of the initial level of nominal prices and the entire state space. We resolve this issue in section 5 below.

The Laffer Curve Let $\varphi_{(\tau_t,y_t)} \equiv \frac{\partial \ln(\tau_t y_t)}{\partial \ln(\tau_t)} = 1 + \frac{\partial \ln(y_t)}{\partial \ln(\tau_t)}$ denote the marginal revenue generated from an increase in taxes. The second term is negative as higher taxes decrease output, so the elasticity of tax revenue with respect to tax rates is less than one. The peak of the "Laffer curve" is where the elasticity, $\varphi_{(\tau_t,y_t)}$, is zero. Similarly, the "slippery side" of the Laffer curve is where the elasticity is negative. Does the model economy exhibit a "Laffer curve" type relation? In this economy $y_t = f(k_t)$, accordingly $\varphi_{(\tau_t,y_t)} = 1 + \frac{\tau_t}{f(k_t)} \frac{\partial f(k_t)}{\partial \tau_t} = 1 + \frac{\tau_t}{f(k_t)} f'(k_t) \frac{dk_t}{d\tau_t}$. We cannot say much about the dynamic Laffer curve. However, we can characterize the Laffer curve near the steady state. It is straightforward to obtain[5] that $\frac{dk^*}{d\tau^*} = \frac{1}{1-\tau^*} \frac{f'(k^*)}{f''(k^*)}$. Thus, the slope of the Laffer curve in the steady state is:

$$\varphi_{(\tau^*)} = 1 + \frac{\tau^*}{1-\tau^*} \xi_k \tag{43}$$

[5] By applying the implicit function theorem on equation (48).

where $\xi_k \equiv \frac{[f'(k^*)]^2}{f(k^*)f''(k^*)} < 0$. This expression implies that a unique maximum of the Laffer curve is obtained at $0 < \overline{\tau} = \frac{1}{1-\xi_k} < 1$.[6]

4.2 General Equilibrium

In equilibrium, markets clear and government liabilities equal household's assets. Using the monetary policy rule (13), the fiscal rule (42), imposing market clearing conditions (16) - (17), and assuming that the elasticity of intertemporal substitution in consumption is constant, the characterization of the general equilibrium of the economy is:

Proposition 7 *In equilibrium with income taxes and finance constraints,*

[6] To get some intuition, consider a Cobb-Douglas production technology $f(k) = k^{\epsilon}$. For this type of production function, $\xi_k = \frac{\epsilon}{\epsilon-1}$ for all k, and the peak of the Laffer curve is where $\overline{\tau} = 1-\epsilon$. Commonly used values of ϵ for developed economies imply that $\overline{\tau}$ exceeds 0.5.

the aggregate dynamics satisfy the following ODE system:

$$\frac{\dot{c}_t}{c_t} = \sigma\left[\frac{1-\tau(k_t,a_t)}{1+\frac{1}{\nu}R(\pi_t)}f'(k_t)-(\rho+\delta)\right] \tag{44}$$

$$\dot{\pi}_t = \frac{\nu+R(\pi_t)}{\alpha}\left\{[R(\pi_t)-\pi_t]-\left[\frac{1-\tau(k_t,a_t)}{1+\frac{1}{\nu}R(\pi_t)}f'(k_t)-\delta\right]\right\} \tag{45}$$

$$\dot{k}_t = f(k_t)-c_t-\delta k_t \tag{46}$$

$$\dot{a}_t = [R(\pi_t)-\pi_t]\,a_t+T_t-\left[\tau(k_t,a_t)+\frac{1}{\nu}R(\pi_t)\right]f(k_t) \tag{47}$$

and equilibrium is defined as follows:

Definition *A perfect-foresight equilibrium with an income tax and finance constraints is a set of sequences* $\{c_t,\pi_t,k_t,a_t,\tau_t,T_t,R_t\}$ *and an initial price level* $P_0>0$ *satisfying (44)-(47) given* $M_0+B_0>0$ *and* $k_0>0$*.*

4.2.1 Steady-State Equilibrium

It follows from equation (44) that in a steady state

$$f'(k^*) = (\rho+\delta)\,\frac{1+\frac{1}{\nu}R^*}{1-\tau^*} \tag{48}$$

where τ^* denotes a long-run income tax rate and R^* is a steady state rate of interest. We can see the distorting effect of income taxes on long run output as the marginal product of capital increases with the tax rate. From equations (45) and (48), R^* must satisfy

$$R^* = \rho + \pi^* \tag{49}$$

where π^* is the long-run rate of inflation. Equation (46) implies that the steady state consumption is

$$c^* = f(k^*) - \delta k^* \tag{50}$$

Finally, it follows from equation (47), that in a steady-state equilibrium government liabilities must satisfy $a^* = \frac{1}{\rho}\left[f(k^*)(\tau^* + \frac{1}{\nu}R^*) - \overline{T}\right]$. Let $\widetilde{a}^* \equiv \frac{a^*}{f(k^*)}$, $\widetilde{T}^* \equiv \frac{\overline{T}}{f(k^*)}$ denote debt to GDP and transfers to GDP in the steady state, respectively, then we obtain that a sustainable debt level must satisfy

$$\widetilde{a}^* = \frac{1}{\rho}\left[\tau^* + \frac{1}{\nu}R^* - \widetilde{T}^*\right] \tag{51}$$

and similar to proposition 2, it is imperative to proclaim three explicit targets. Specifically,

Corollary 3 *Where government proclaims only fiscal targets such as* $\left\{\tau^*, \widetilde{a}^*, \widetilde{T}^*\right\}$, *the inflation target is implied according to* $\pi^* = \nu\left(\rho\widetilde{a}^* + \widetilde{T}^* - \tau^*\right) - \rho$ *and lump-sum transfers are then set according to* $\overline{T} = \widetilde{T}^* f(k^*)$.

Corollary 4 *Where government proclaims that the explicit set of targets is* $\left\{\tau^*, \pi^*, \overline{T}\right\}$, $\widetilde{a}^*$ *is implied directly by condition (51).*

and in the rest of the paper we assume that the government explicitly proclaims $\left\{\tau^*, \pi^*, \overline{T}\right\}$.

4.2.2 Equilibrium Dynamics

Price level determination Solving equation (47), and letting $t \to \infty$ yields that market equilibrium requires intertemporal government budget balance:

Lemma 2 *In equilibrium*

$$0 = \lim_{t\to\infty} e^{-\int_0^t [R(\pi_s)-\pi_s]ds} a_t = a_0 - \int_0^{\infty} e^{-\int_0^t [R(\pi_s)-\pi_s]ds} \left\{f(k_t)\left[\tau_t + \tfrac{1}{\nu}R(\pi_t)\right] - \overline{T}\right\} dt$$

The arguments for justifying Lemma 2 are identical to those of Lemma 1. Note that substituting the fiscal rule (42) into (47) yields that government liabilities evolve according to:

$$\dot{a}_t = \left[R(\pi_t) - \pi_t - \gamma\frac{f(k_t)}{a^*}\right]a_t - f(k_t)\left[\tau^* + \beta\frac{f(k_t) - f(k^*)}{f(k^*)} - \gamma + \frac{1}{\nu}R(\pi_t)\right] + \overline{T} \tag{52}$$

Solving equation (52) for a_t we obtain that:

$$Q_t a_t = a_0 - \int_0^t Q_s \left\{f(k_s)\left[\tau^* + \beta\frac{f(k_s) - f(k^*)}{f(k^*)} - \gamma + \frac{1}{\nu}R(\pi_s)\right] - \overline{T}\right\} ds \tag{53}$$

where $Q_t \equiv e^{-\int_0^t \left[R(\pi_s) - \pi_s - \gamma\frac{f(k_s)}{a^*}\right]ds}$. Letting $t \to \infty$ and rearranging we obtain that:

$$\lim_{t\to\infty} e^{-\int_0^t [R(\pi_s)-\pi_s]ds} a_t \tag{54}$$

$$= \lim_{t\to\infty} \left[\frac{a_0 - \int_0^t Q_s \left\{ f(k_s) \left[\tau^* + \beta \frac{f(k_s)-f(k^*)}{f(k^*)} - \gamma + \frac{1}{\nu} R(\pi_s) \right] - \overline{T} \right\} ds}{e^{\int_0^t \gamma \frac{f(k_s)}{a^*} ds}} \right]$$

and equation (54) has the following implications:

Proposition 8 *The equilibrium price level is determined to satisfy*

$$\frac{B_0+M_0}{P_0} = \int_0^\infty Q_s \left\{ f(k_s) \left[\tau^* + \beta \frac{f(k_s)-f(k^*)}{f(k^*)} - \gamma + \frac{1}{\nu} R(\pi_s) \right] - \overline{T} \right\} ds$$

if and only if

a) $\gamma < \rho \widetilde{a}^*$

b) *there is a unique trajectory that converges to equilibrium*

Proposition 8 is the counterpart to Proposition 3 where the fiscal authority has no access to lump-sum taxation. It shows how nominal prices are determined if the fiscal authority lets it's liabilities grow at a rate greater than the real interest rate. The contribution here is that fiscal policy should be considered active where the response of income tax rate to government debt is less than $\rho\widetilde{a}^{*}$. This result has an important implication on the choice of policy targets. The debt-to-GDP target ratio now influences the threshold under which fiscal policy is considered active. When the government operates a passive fiscal rule, where $\gamma > \rho\widetilde{a}^{*}$, the level of nominal prices cannot be determined according to Proposition 8. Policy stances of this type imply that the government ensures that its liabilities will converge back to the target and that for any price level fiscal solvency is ensured by the fiscal policy. In this case P_0 is determined to satisfy $\frac{M_0}{P_0} = \frac{1}{\nu}(c_0 + I_0)$.

Transitional Dynamics Let $\dot{x}_t = g(x_t)$ denote the system of equations (44)-(47) then a linear approximation to $\dot{x}_t = g(x_t)$ near the steady state is obtained through the system

$$\dot{x}_t = B \times (x_t - \overline{x}) \tag{55}$$

where

$$B \equiv \begin{bmatrix} 0 & -\frac{\sigma\alpha\widetilde{c}^*(\rho+\delta)}{\nu+R^*}f^* & -\frac{\sigma\widetilde{c}^*}{1+\frac{1}{\nu}R^*}f_k^{*2}\left(\beta+\frac{\tau^*}{1-\varphi^*}\right) & -\gamma\frac{\sigma(\rho+\delta)}{1-\tau^*}\frac{\widetilde{c}^*}{\widetilde{a}^*} \\ 0 & \rho+\delta+(\nu+R^*)\frac{\alpha-1}{\alpha} & \frac{\nu}{\alpha}\frac{f_k^{*2}}{f^*}\left(\beta+\frac{\tau^*}{1-\varphi^*}\right) & \gamma\frac{\nu}{\alpha}\frac{f_k^*}{f^*}\frac{1}{\tilde{a}^*} \\ -1 & 0 & f_k^*-\delta & 0 \\ 0 & f^*\left[(\alpha-1)\tilde{a}^*-\frac{\alpha}{\nu}\right] & -f_k^*\left[\frac{1}{\nu}R^*+\beta+\tau^*\right] & \rho-\frac{\gamma}{\tilde{a}^*} \end{bmatrix} \tag{56}$$

Let $r_i \quad i=1,..,4$ denote the eigenvalues of B, then we obtain that:

$$r_1r_2r_3r_4 = -\left[\widetilde{c}^*\nu\sigma\rho f_k^{*2}\right]\frac{\alpha-1}{\alpha}\left[\beta+\frac{\tau^*}{1-\varphi^*}+\frac{\gamma}{\rho\tilde{a}^*}\left(\frac{1}{\nu}R^*-\frac{\tau^*\varphi^*}{1-\varphi^*}\right)\right] \tag{57}$$

$$r_1+r_2+r_3+r_4 = 2\rho+(\nu+R^*)\frac{\alpha-1}{\alpha}+(\rho+\delta)\frac{1+\frac{1}{\nu}R^*}{1-\tau^*}-\frac{\gamma}{\tilde{a}^*} \tag{58}$$

and Proposition 9 follows directly from equation (57),

Proposition 9 *The steady state $\overline{x}$ is hyperbolic if and only if*

$\frac{\alpha-1}{\alpha}\left[\beta+\frac{\tau^*}{1-\varphi^*}+\frac{\gamma}{\rho\tilde{a}^*}\left(\frac{1}{\nu}R^*-\frac{\tau^*\varphi^*}{1-\varphi^*}\right)\right]\neq 0.$

Furthermore, given that (k_t, a_t) are predetermined we obtain that:

Proposition 10 *A necessary condition for determinacy of equilibrium is*

$$\frac{\alpha-1}{\alpha}\left[\beta+\frac{\tau^*}{1-\varphi^*}+\frac{\gamma}{\rho\tilde{a}^*}\left(\frac{1}{\nu}R^*-\frac{\tau^*\varphi^*}{1-\varphi^*}\right)\right]<0.$$

Proposition 10 points out that a monetary policy such that $\alpha > 1$ must interact with a fiscal policy such that $\beta+\frac{\tau^*}{1-\varphi^*}+\frac{\gamma}{\rho\tilde{a}^*}(\frac{1}{\nu}R^*-\frac{\tau^*\varphi^*}{1-\varphi^*})<0$, and a monetary policy such that $\alpha < 1$ must interact with a fiscal policy such that $\beta+\frac{\tau^*}{1-\varphi^*}+\frac{\gamma}{\rho\tilde{a}^*}(\frac{1}{\nu}R^*-\frac{\tau^*\varphi^*}{1-\varphi^*})>0$. Note however that these are necessary conditions. I next discuss policy regimes from which we can derive sufficient conditions for equilibrium determinacy.

5 Monetary-Fiscal Interactions with an Active Fiscal Policy

In what follows I focus on a baseline regime for which it is possible to obtain sufficient conditions. I then perturb the baseline regime so as to approximate the general case.

5.1 A Baseline Stabilizing Interaction

Consider a fiscal policy where $\gamma = 0$. According to this rule, the income tax rate responds only to deviations of output from its long run level. Since we assume that lump-sum transfers are constant, this also implies that the tax rate responds to deviations of the primary deficit from its target. Note that under this fiscal rule $\gamma = 0 < \rho\widetilde{a}^*$, and according to Proposition 3 it satisfies a necessary condition for price level determination. However, fiscal solvency also depends on whether the monetary-fiscal regime induces a unique trajectory. Linearizing $\dot{x}_t = g_{[\gamma=0]}(x_t)$ the system becomes

$$\dot{x}_t = B_{[\gamma=0]} \times (x_t - \overline{x}) \tag{59}$$

where

$$B_{[\gamma=0]} = \begin{bmatrix} \widehat{B_1} & 0 \\ \widehat{B_2} & \rho \end{bmatrix}$$

and where $\widehat{B_1}$ is the upper left 3×3 submatrix of B, $\widehat{B_2}$ is the 1×3 row vector $\{B_{4,1}, B_{4,2}, B_{4,3}\}$ and the zero stands for a 3×1 vector of zeros. Examining $B_{[\gamma=0]}$, the dynamics of (c, π, k) are independent of government liabilities. This feature has two implications: (a) one eigenvalue of the (c, π, k, a) system is $\rho > 0$; (b) the remaining three eigenvalues are determined by $\widehat{B_1}$ so that the dynamics of (c, π, k) are completely determined by $\widehat{B_1}$. It is straightforward to show that the three remaining eigenvalues satisfy:

$$r_1 r_2 r_3 = -\left[\widetilde{c}^* \nu \sigma \rho f_k^{*2}\right] \frac{\alpha - 1}{\alpha} \left[\beta + \frac{\tau^*}{1 - \varphi^*}\right] \tag{60}$$

$$r_1 + r_2 + r_3 = \rho + (\nu + R^*) \frac{\alpha - 1}{\alpha} + (\rho + \delta) \frac{1 + \frac{1}{\nu} R^*}{1 - \tau^*} \tag{61}$$

Proposition 11 *If fiscal policy targets only the primary deficit, then a sufficient condition for equilibrium determinacy is that the monetary-fiscal regime act according to:* $\alpha < \frac{1}{1 + \frac{\rho}{\nu + R^*} + \frac{\rho + \delta}{\nu(1 - \tau^*)}} < 1$ *and* $\beta > -\frac{\tau^*}{1 - \varphi^*}$.

In the terminology of the fiscal theory of prices, this regime is often referred to as regime F under which the government can issue nominal bonds to finance primary deficits. In simple models, where government has access to lump sum taxation and output is exogenous, policy actions are transmitted through wealth effects.

Introducing capital and income taxes into the model gives rise to an arbitrage channel. Households optimize on their wealth portfolio which consists of productive capital, nominal bonds, and cash. Bonds yield a real return that equals the nominal rate of interest minus inflation whereas an increment to the stock of capital yields a real return that is distorted by income tax and a liquidity requirement. At the private level bonds and capital are perfect substitutes. Accordingly, equilibrium implies an intertemporal no-arbitrage between bonds and capital. The lack of arbitrage implies that the difference between the real return on bonds and the distorted marginal product of capital must equal the expected change in inflation.

Having two channels, any change of the income tax rate or the nominal rate of interest induces an arbitrage effect as well as a wealth effect. This complication can cause multiplicity of equilibria, which according to proposition 8 is detrimental to the valuation of government debt.

To prevent multiple equilibria, fiscal policy should respond to output such that $\beta + \frac{\tau^*}{1-\varphi^*} > 0$. Note that the evaluation of $\frac{d \ln \left[\frac{1-\tau(k_t, a_t)}{1+\frac{1}{\nu} R(\pi_t)} f'(k_t) \right]}{d \ln f(k_t)}$ near the steady state is $-\frac{1}{1-\tau^*} \left[\beta + \frac{\tau^*}{1-\varphi^*} \right]$. That is, the evolution of after tax marginal product of capital along an equilibrium trajectory is sensitive to fiscal responses to output. When fiscal policy exhibits responses to output such that $\beta + \frac{\tau^*}{1-\varphi^*} < 0$, the after tax marginal product of capital becomes positively associated with output. This indicates a flawed policy and the intuition is the following: start from a steady state equilibrium, and suppose that the future return on capital is expected to increase. Without distorting taxes, indeterminacy cannot occur since a higher capital stock is associated with a lower rate of return under constant returns to scale. However, a policy where $\beta + \frac{\tau^*}{1-\varphi^*} < 0$ causes the after tax return on capital to rise even further thus validating agents' expectations and any such trajectory is consistent with equilibrium. By contrast, a stance such that $\beta + \frac{\tau^*}{1-\varphi^*} > 0$ reduces higher anticipated returns on capital from belief-driven expansions thus preventing expectations from becoming self fulfilling.

5.2 Perturbations to the Parameter Space

Consider now complex fiscal rules that exhibit $\gamma \neq 0$. In what follows I claim that for small perturbations of γ the system is structurally stable. Consider the system $\dot{x_t} = g_{[\gamma]}(x_t)$ where $\gamma = 0 \pm \varepsilon$, $\varepsilon > 0$, then a linearization reads

$$\dot{x_t} = \left[B_{[\gamma=0]} + \varepsilon\Delta\right] \times (x_t - \overline{x}) \tag{62}$$

where

$$\Delta \equiv \begin{bmatrix} 0 & 0 & 0 & -\frac{\sigma(\rho+\delta)}{1-\tau^*}\frac{\widetilde{c}^*}{\widetilde{a}^*} \\ 0 & 0 & 0 & \frac{\nu}{\alpha}\frac{f_k^*}{f^*}\frac{1}{\widetilde{a}^*} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\widetilde{a}^*} \end{bmatrix}$$

Since $B_{[\gamma=0]}$ is invertible, by implicit function theorem, $\dot{x}_t = g_{[\gamma]}(x_t)$ continues to have a unique solution $\overline{\overline{x}} = \overline{x} + O(\varepsilon)$ near $\overline{x}$ for sufficiently small ε. Moreover, since we restrict the admissible monetary-fiscal regimes to a set that satisfies proposition 10, we ensure that $B_{[\gamma=0]} + \varepsilon\Delta$ is invertible which implies that $\overline{\overline{x}} = \overline{x}$ is the unique solution to equation (62). Furthermore, since the matrix of the linearized system $B_{[\gamma=0]} + \varepsilon\Delta$ has eigenvalues that depend continuously on ε, no eigenvalues can cross the imaginary axis if ε remains small with respect to the magnitude of the real parts of the eigenvalues of $B_{[\gamma=0]}$. Thus, the perturbed system (62) has a unique fixed point with eigenspaces and invariant manifolds of the same dimensions as those of the unperturbed system and which are $\varepsilon-$close locally in position and slope to the unperturbed manifolds. The magnitude of perturbation that preserves the topological equivalence between the perturbed and the unperturbed systems is described in proposition 12.

Proposition 12 *Consider a regime that induces equilibrium near a hyperbolic steady state* $\overline{x}$ *where fiscal policy targets the primary deficit. Assume that its policy stances are* $\overline{\alpha}$*,*$\overline{\beta}$ and that $\overline{\gamma} = 0$*. Then perturbations to* γ *in the neighborhood of* $\overline{\gamma} = 0$ *do not change the phase portrait of* $\overline{x}$ *as long as* γ *is not perturbed until its bifurcation point. Specifically, if* $\overline{\alpha} < \frac{1}{1+\frac{\rho}{\nu+R^*}+\frac{\rho+\delta}{\nu(1-\tau^*)}}$ *and* $\overline{\beta} > -\frac{\tau^*}{1-\varphi^*}$ *then a complex monetary fiscal regime will also induce equilibrium as long as the multiple of eigenvalues in the perturbed system does not change signs. That is, it satisfies* $\frac{\overline{\alpha}-1}{\overline{\alpha}}\left[\overline{\beta} + \frac{\tau^*}{1-\varphi^*} + \frac{\gamma}{\rho\tilde{a}^*}(\frac{1}{\nu}R^* - \frac{\tau^*\varphi^*}{1-\varphi^*})\right] < 0.$

Table 1 summarizes the various regimes that induce an equilibrium under the active fiscal stance.

Table 1 - Stabilizing Responses to Public Debt under Active Fiscal Policy

[in all regimes $\alpha < \frac{1}{1+\frac{\rho}{\nu+R^*}+\frac{\rho+\delta}{\nu(1-\tau^*)}}$ *and* $\beta + \frac{\tau^*}{1-\varphi^*} > 0$*]*

Regime	Revenues elasticity	Response to Gov. Liabilities
Deficit Targeting		$\gamma = 0$
Complex Fiscal Rule	$\frac{1}{\nu}R^* < \frac{\tau^*\varphi^*}{1-\varphi^*}$	$\gamma < Min[\rho\widetilde{a}^*, \frac{\beta+\frac{\tau^*}{1-\varphi^*}}{\frac{\tau^*\varphi^*}{1-\varphi^*}-\frac{1}{\nu}R^*}\rho\widetilde{a}^*]$
Complex Rule near the peak of Laffer Curve	$\frac{1}{\nu}R^* > \frac{\tau^*\varphi^*}{1-\varphi^*}$	$-\frac{\beta+\frac{\tau^*}{1-\varphi^*}}{\frac{1}{\nu}R^*-\frac{\tau^*\varphi^*}{1-\varphi^*}}\rho\widetilde{a}^* < \gamma < \rho\widetilde{a}^*$

5.3 Computation

We obtained so far that if the monetary -fiscal policy interaction is stabilizing in the sense that it induces a unique trajectory we can compute the entire equilibrium path. Given two initial conditions the model becomes a continuous-time variant of Blanchard and Kahn (1980) and impulse responses can then be computed according to Sims (2002). However, computing the entire equilibrium involves a simultaneous determination of the initial level of nominal prices and the entire state space. Note Proposition 8. In the fiscal theoretic pricing of government liabilities, a policy shock causes a reevaluation of the existing nominal stock of bond and money. The after-shock value equals the present value of the sum future surpluses. However, to compute this stream of surpluses we need the initial value of non capital wealth. Thus, the initial value of government liabilities depends on the entire equilibrium which depends on the initial value of government liabilities. In what follows we formalize this issue and its resolution.

Corresponding to the autonomous system $\dot{x_t} = g(x_t)$ which denotes the system of equations (44)-(47) and where $g : W \rightarrow E$, there are maps $\phi : \Omega \rightarrow W$ where $(t, a, k, \tau, \pi) \in \Omega$ satisfying $\dot{\phi_t} = g(\phi_t)$, ϕ is defined by letting $t \longrightarrow \phi_t(a, k, \tau, \pi) = \phi(t, a, k, \tau, \pi)$ be the maximal solution curve taking 0 to (a, k, τ, π). We obtained that given that the monetary-fiscal policy interaction is according to Proposition 12 and given two initial values a_0, k_0, there is a unique homeomorphism ϕ_t on $[0, \infty)$ sending 0 to $(a_0, k_0, \tau^*, \pi^*)$.

Note Proposition 8. Let $\psi : W \rightarrow \Re_+$ correspond to the mapping of the flow ϕ_t into a positive real number that indicates the real value of government liabilities. Specifically, $\psi \equiv \int\limits_0^\infty Q_s \left\{ f(k_s) \left[\tau^* + \beta \frac{f(k_s) - f(k^*)}{f(k^*)} - \gamma + \frac{1}{\nu} R(\pi_s) \right] - \overline{T} \right\} ds.$ Let F denote the composition $\psi \circ \phi$ then $F : \Omega \rightarrow \Re_+$ and $F(a_t, k_t, \tau^*, \pi^*)$ returns the valuation of government liabilities given that the inflation target is

π^* the tax rate target is τ^* and the state variables are (a_t, k_t). The economy is in equilibrium if and only if $\forall t \in [0, \infty)$ $a_t = F(a_t, k_t, \tau^*, \pi^*)$ and we can claim to have pinned down the initial price level when $a_0 = F(a_0, k_0, \tau^*, \pi^*)$.

Note that $a^* = F(a^*, k^*, \tau^*, \pi^*) = \frac{1}{\rho}\left[f(k^*)(\tau^* + \frac{1}{\nu}R^*) - \overline{T}\right]$ which implies that where inflation and taxes are on target and the capital stock is at its long run level the valuation of government liabilities is also on target. In terms of the composition $\psi \circ \phi$, the flow that solves eq. (44)-(47) is stationary since the economy is in a steady state equilibrium. Thus, the mapping ψ brings the present value of constant surpluses, and this valuation is exactly the value of government liabilities in the steady state.

We now proceed to obtaining sufficient conditions for the existence of an equilibrium initial valuation such that $a_0 = F(a_0, k_0, \tau^*, \pi^*)$ and to a program that obtains this value. Consider an economy in a steady state equilibrium where $a^* = F(a^*, k^*, \tau^*, \pi^*)$. Any shift from the steady state implies a revaluation of government liabilities which implies $dF \neq 0$. Note that $dF = \left[\frac{\partial F}{\partial a}\right]_{a^*} da + \left[\frac{\partial F}{\partial k}\right]_{k^*} dk + \left[\frac{\partial F}{\partial \tau}\right]_{\tau^*} d\tau + \left[\frac{\partial F}{\partial \pi}\right]_{\pi^*} d\pi$ which implies that a change in the valuation of government liabilities can arrive either from a change in policy targets - for example embracing a new target set (τ^{**}, π^{**}) instead of (τ^*, π^*) - or due to a shock to one of the state variables. As we rule out shocks to the stock of capital, capital is predetermined and at the moment that a policy innovation is realized $dk = 0$. Hence a change in policy targets has two effects: First, it induces a reevaluation of government liabilities. Second, it sends the economy to a transition towards a new steady state where the equilibrium trajectory depends on two initial conditions - the stock of capital and the revalued government debt. Let a_0 denote the equilibrium initial valuation of government liabilities then it reads $a_0 = \digamma(a_0, k^*, \tau^{**}, \pi^{**})$ and the total differential reads $d\digamma - \left[\left[\frac{\partial \digamma}{\partial \tau}\right]_{\tau^*}(\tau^{**} - \tau^*) + \left[\frac{\partial \digamma}{\partial \pi}\right]_{\pi^*}(\pi^{**} - \pi^*)\right] = \left[\frac{\partial \digamma}{\partial a}\right]_{a^*} da$ where $da = a_0 - a^*$. Also note that $\mathrm{Sign}(d\digamma) = \mathrm{Sign}\left[\left[\frac{\partial \digamma}{\partial \tau}\right]_{\tau^*}(\tau^{**} - \tau^*) + \left[\frac{\partial \digamma}{\partial \pi}\right]_{\pi^*}(\pi^{**} - \pi^*)\right]$ which states that once the economy is in the steady state, a revaluation of government liabilities arrives from innovations to policy targets.

Lemma 3 *Assume that* $\mathrm{Sign}(d\digamma) = \mathrm{Sign}\left[\left[\frac{\partial \digamma}{\partial \tau}\right]_{\tau^*}(\tau^{**} - \tau^*) + \left[\frac{\partial \digamma}{\partial \pi}\right]_{\pi^*}(\pi^{**} - \pi^*)\right]$ *then* $\widetilde{\digamma}(a) \equiv \digamma(a, k^*, \tau^{**}, \pi^{**})$ *is Lipschitz on* $[a^*, a^{**}]$.

Proof Note that

$$\left|d\widetilde{\digamma}\right| \leqslant \left|\left[\frac{\partial \digamma}{\partial a}\right]_{a^*}\right| |da| \Longleftrightarrow \mathrm{Sign}(d\digamma) = \mathrm{Sign}\left[\left[\frac{\partial \digamma}{\partial \tau}\right]_{\tau^*}(\tau^{**} - \tau^*) + \left[\frac{\partial \digamma}{\partial \pi}\right]_{\pi^*}(\pi^{**} - \pi^*)\right]$$

Let the Lipschitz constant be $\Bbbk \equiv \left|\left[\frac{\partial \digamma}{\partial a}\right]_{a^*}\right|$, and this concludes the proof.

Note that $a^* = \digamma(a^*, k^*, \tau^*, \pi^*)$ and that by construction $\left[\frac{\partial \digamma}{\partial a}\right]_{a^*} = 1.$ This characteristic of the model means that any change in the economy that does not imply a change in the after tax marginal productivity is entirely absorbed by a reevaluation of government liabilities. For example, a shock to the nominal level of public debt or a permanent shock to transfers that are not accompanied by a change in inflation target or tax rate target will not cause a transition to a new steady state. The only effect will be an adjustment of nominal prices so as to revalue government liabilities. This characteristic has an important implication: where the Lipschitz constant, $\Bbbk$, equals 1 we obtain that $\widetilde{\digamma}(a)$ is a non-expanding map. Let C be a closed convex subset of a Banach space E. A mapping T on C is called a non-expanding mapping if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $\{\overline{x}\}$ the set of fixed points of T. If C is compact then Schauder's fixed point theorem yields that $\{\overline{x}\}$ is non empty. Obtaining members of $\{\overline{x}\}$ is then feasible via a Krasnoselski and Mann's type sequence. Krasnoselski's Theorem is the following result:

Krasnoselski's Theorem If C is a convex, bounded subset of a uniformly convex Banach space and if T is a mapping of C into a compact subset of C such that $\|T(x) - T(y)\| \leq \|x - y\|$, then the sequence obtained by choosing x_1 in C and defining $x_{n+1} = \frac{1}{2}\left[x_n + T(x_n)\right]$ converges to some z in C and $T(z) = z$.

Note that we study $\widetilde{F}(a)$ of which the domain is a closed interval of the real line. Bailey (1974) gave a proof for the case in which C is a closed interval of the real line. Bailey's Theorem is the following:

Bailey's Theorem If T takes $[a, b]$ into itself and $\|T(x) - T(y)\| \leq \|x - y\|$, then the sequence obtained by choosing x_1 in $[a, b]$ and defining $x_{n+1} = \frac{1}{2}\left[x_n + T(x_n)\right]$ converges to some z in $[a, b]$ and $T(z) = z$.

Mann's (1953) theorem states that if T is a continuous function that takes $[a, b]$ into itself and having a unique fixed point, z in $[a, b]$, then the Krasnoselski-Bailey sequence converges to z for all choices of x_1 in $[a, b]$. This leads us to the following proposition:

Proposition 13 An equilibrium with an active fiscal policy exists and is unique if and only if the monetary-fiscal policy interaction parameters are chosen according to proposition 12.

Proof The proof follows directly from Proposition 8, Proposition 12, Lemma 3, and Bailey's Theorem.

6 Monetary-Fiscal Interactions with a Passive Fiscal Policy

6.1 A Baseline Stabilizing Interaction

In what follows I focus on a baseline regime for which it is possible to obtain sufficient conditions for the existence and uniqueness of an equilibrium where income tax rate responds to a percentage deviation of government liabilities from it's long run level by more than $\rho\widetilde{a}^*$. We can then perturb the baseline regime to obtain approximation to the general case. Consider a fiscal policy where $\gamma > \rho\widetilde{a}^*$. Also, Proposition 10 provides the necessary conditions with respect to the set (α, β, γ) for a unique determination of an equilibrium trajectory. However, observing Table A.1 we can notice that the necessary condition is satisfied also by fixed points with indices 0 and 4. To rule out these possibilities we require that the sum of eigenvalue equals zero. Thus, policy parameters that satisfy Proposition 10 and at the same time equate the right hand side of equation (58) to zero ensure that the monetary-fiscal interaction brings about an equilibrium with the "right" dimensions. This set of parameters is a starting point from which we can perturb the system.

Consider an active monetary-passive fiscal regime, where $\overline{\alpha} > 1$ and

$\overline{\gamma} = \tilde{a}^* \left[2\rho + (\nu + R^*)\frac{\overline{\alpha}-1}{\overline{\alpha}} + (\rho+\delta)\frac{1+\frac{1}{\nu}R^*}{1-\tau^*} \right] > \rho\widetilde{a}^*$. We obtained $\overline{\gamma}$ by equating the right hand side of eq. (58) to zero. It is imperative at this point to make assumptions with respect to the elasticity of tax revenues. We will assume that the economy is neither at the peak nor it is on the "wrong side" of its Laffer curve. This assumption is formalized by $\frac{\tau^*\varphi^*}{1-\varphi^*} - \frac{1}{\nu}R^* > 0$. Then, for $\overline{\alpha}$ and $\overline{\gamma}$, for any $\overline{\beta}$ that is in the range $0 < \overline{\beta} + \frac{\tau^*}{1-\varphi^*} < \frac{\overline{\gamma}}{\rho\tilde{a}^*}(\frac{\tau^*\varphi^*}{1-\varphi^*} - \frac{1}{\nu}R^*)$ we can be certain that the fixed point has the desired phase portrait i.e. that $ind(\overline{x}) = 2$. Thus, an active-monetary Passive-fiscal regime with policy responses $\overline{\alpha}, \overline{\beta}$, and $\overline{\gamma}$, induces a unique trajectory that converges to the steady state.

6.2 Perturbations to the Parameter Space

Consider now the system $\dot{x}_t = g_{[\gamma]}(x_t)$ where $\gamma = \overline{\gamma} \pm \varepsilon$, $\varepsilon > 0$, then a linearization reads $\dot{x}_t = \left[B_{[\gamma=\overline{\gamma}]} + \varepsilon\Delta \right] \times (x_t - \overline{x})$. As demonstrated in the previous section, $\overline{x}$, the fixed point of the unperturbed system is by construction the fixed point of the perturbed system. Our next step is to search for values of ε such that preserves the topological equivalence between the perturbed and the unperturbed systems. This magnitude is described in proposition 14.

Proposition 14 *Consider an active-monetary passive-fiscal regime where elasticities of tax revenues are in a range* $\frac{\tau^*\varphi^*}{1-\varphi^*} - \frac{1}{\nu}R^* > 0$ *and policy parameters are* $\overline{\alpha} > 1$, $\overline{\gamma} = \tilde{a}^*\left[2\rho + (\nu + R^*)\frac{\overline{\alpha}-1}{\overline{\alpha}} + (\rho+\delta)\frac{1+\frac{1}{\nu}R^*}{1-\tau^*}\right] > \rho\widetilde{a}^*$, and $0 < \overline{\beta} + \frac{\tau^*}{1-\varphi^*} < \frac{\overline{\gamma}}{\rho\tilde{a}^*}(\frac{\tau^*\varphi^*}{1-\varphi^*} - \frac{1}{\nu}R^*)$. *Then this regime induces a hyperbolic equilibrium* $\overline{x}$ *with* $ind(\overline{x}) = 2$*. Right perturbations,* $\gamma = \overline{\gamma} + \varepsilon$, $\varepsilon > 0$, *do not change the phase portrait of* $\overline{x}$*.*

Note that this is a sufficient condition and not necessary. This means that there are left perturbations that preserve the desired topological equivalence however their magnitude depends on the choice of $\overline{\beta}$ whereas right perturbations preserve topological equivalence as long as the economy remains on the "correct" side of its Laffer curve. An important implication of this exercise is that when the economy is not on the "correct" side of its Laffer curve we have $\frac{\tau^*\varphi^*}{1-\varphi^*} - \frac{1}{\nu}R^* \leq 0$. As a result we cannot obtain a baseline active monetary-passive fiscal regime that satisfies Proposition 10.

Proposition 15 *At the fiscal limit, i.e. where* $\frac{\tau^*\varphi^*}{1-\varphi^*} - \frac{1}{\nu}R^* \leq 0$ *there is no stabilizing regime with a passive fiscal policy.*

6.3 Computation

Computing equilibrium where fiscal policy is passive is straightforward:

I. $M_0, B_0, k_0,$ are given.

II. The initial price level in pinned down by solving $\frac{M_0}{P_0} = \frac{1}{\nu}(c_0 + I_0) = \frac{1}{\nu} f(k_0)$.

III. Having P_0 we obtain $a_0 = \frac{B_0 + M_0}{P_0}$

IV. Having (a_0, k_0) we solve the linear rational expectation model in eq. (55) according to Sims (2002).

7 Concluding Remarks

We have shown how income taxes complicate conventional results in macroeconomic models with interactions between monetary policy and fiscal policy. Where the government issues nominal debt that is denominated in home currency, the initial real value of financial wealth is missing. As a result the initial level of prices must be determined. A passive fiscal policy, that forces tax revenues to grow at a faster rate than the rate of growth of government debt, induces an equilibrium where the price level is determined by balance between money supply and money demand. This policy pin points an initial level of prices which induces an initial valuation for government debt and this suffices to calculate the rational expectations equilibrium according to Blanchard and Kahn. However, passive fiscal policies are not always at hand either because there is no access to lump-sum taxation or because the government is approaching its fiscal limit. In such situations the initial level of prices is pin pointed according to the fiscal theory of prices and the initial value of government debt is a solution to a fixed point problem. This paper specifies an algorithm for evaluating government debt where fiscal policy is active. A striking result is that with lump-sum taxation the types of monetary-fiscal interactions are characterized by the response of the fiscal authority to the secondary deficit whereas with distorting taxation the types of monetary-fiscal interactions are characterized by the response of the fiscal authority to the primary deficit.

Appendix

Proof of Proposition 4 Assume that $(\alpha-1)(\gamma-\rho)\neq 0$ then the multiple of eigenvalues is non zero which indicates that there is no zero eigenvalue. Assume now that $(\alpha-1)(\gamma-\rho)=0$ then either $\alpha=1$ or $\gamma-\rho$. If $\alpha=1$ and $\gamma\neq\rho$

$$A_{[\alpha=1]}\equiv\begin{bmatrix} 0 & -\frac{\sigma\alpha\widetilde{c}^*(\rho+\delta)}{\nu+R^*}f^* & -\frac{\sigma\widetilde{c}^*}{1+\frac{1}{\nu}R^*}\frac{f_k^{*2}}{\xi_k} & 0 \\ 0 & \rho+\delta & \frac{\nu}{\alpha}\frac{f_k^{*2}}{f^*\xi_k} & 0 \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{bmatrix}$$

where $A_{i,j}$ $i,j=1,..4$ are components of A specified in eq. (29), respectively. Where $\alpha=1$ the central bank holds the real rate of interest constant. This policy induces a linear dependence between the first and the second row of $A_{[\alpha=1]}$. Specifically, the first row is a multiplication of the second row by $-\frac{\sigma\alpha\widetilde{c}^*}{\nu+R^*}f^*$. Consequently $A_{[\alpha=1]}$ is singular.

Assume $\alpha\neq 1$ and $\gamma=\rho$ than clearly $A_{[\gamma=\rho]}$ is singular.

Proof of Proposition 5

Preliminaries According to Proposition 4 if $(\alpha-1)(\gamma-\rho)>0$ the steady state is hyperbolic. The rest of the proof is based on the following Theorem and Definition.

Definition [Hirsch and Smale (1976) Chap.16] *Let $\overline{x}$ be a hyperbolic equilibrium, that is, the eigenvalues of $Df(\overline{x})$ have nonzero real parts. In this case, the index $ind(\overline{x})$ of $\overline{x}$ is the number of eigenvalues (counting multiplicities) of $Df(\overline{x})$ having negative real parts.*

The Stable Manifold Theorem

[Guckenhaimer and Holmes (1983) Theorem 1.3.2] *Suppose that $\dot{x} = f(x)$ has hyperbolic fixed point $\overline{x}$. Then there exists local stable and unstable manifolds $W^s_{loc}(\overline{x}), W^u_{loc}(\overline{x})$, of the same dimensions n_s, n_u as those of the eigenspaces E^s, E^u of the linearized system, respectively, and tangent to E^s, E^u at $\overline{x}$. $W^s_{loc}(\overline{x}), W^u_{loc}(\overline{x})$ are as smooth as the function f.*

Thus, the index of a hyperbolic fixed point is the dimension of the stable manifold. Given that we have two predetermined variables equilibrium $\overline{x}$ is determinate if and only if $ind(\overline{x}) = 2$. In what follows I prove Proposition 5.

Proof Note equation (30), the structural parameters ρ, σ are positive. Hence the sign of the right hand side of equation (30) is determined by the sign of $(\alpha - 1)(\gamma - \rho)$

Table A.1: Index and equilibria in a four dimensional vector space with two predetermined variables

Sign(rr_1)	Sign(rr_2)	Sign(rr_3)	Sign(rr_4)	det(A)	Trace(A)	Index	Equilibrium
$+$	$+$	$+$	$+$	>0	>0	0	no-equilibrium
$-$	$+$	$+$	$+$	<0	$\gtreqless 0$	1	no-equilibrium
$-$	$-$	$+$	$+$	>0	$\gtreqless 0$	2	unique
$-$	$-$	$-$	$+$	<0	$\gtreqless 0$	3	multiple
$-$	$-$	$-$	$-$	>0	<0	4	multiple

rr_i denotes the real part of eigenvalue r_i. In our model equilibrium is determinate only where $ind(\overline{x}) = 2$. Where $ind(\overline{x}) = 0, 1$ the system has "too many" unstable roots. That is, there are fewer stable roots than predetermined variables and no convergent solution exists for arbitrary initial value of the predetermined variable[7]. Where $ind(\overline{x}) = 3, 4$ there are more stable roots than predetermined variables. In these cases, the transversality condition that the solution be convergent no longer suffices to ensure a unique solution, and thus, with no additional linear boundary conditions equilibrium

[7] This issue is discussed in detail in Blanchard and Kahn (1980) and in Buiter (1984).

is indeterminate.

Note that $ind(\overline{x}) = 2 \Rightarrow \det(A) > 0 \Rightarrow (\alpha - 1)(\gamma - \rho) > 0$. QED.

Proof of Proposition 6

Consider an active fiscal stance, i.e. $\gamma < \rho$.

In this regime the eigenvalue $\rho - \gamma$ is positive. Hence, monetary policy must bring about two stable eigenvalues via $\widehat{A}$ the upper left 3×3 submatrix of A. Note Table A.2

Table A.2: Index and equilibria in a three dimensional vector space with two predetermined variables

Sign(rr_1)	Sign(rr_2)	Sign(rr_3)	$\det(\widehat{A})$	$\mathrm{tr}(\widehat{A})$	Index	Equilibrium
$+$	$+$	$+$	> 0	> 0	0	no-equilibrium
$-$	$+$	$+$	< 0	$\gtrless 0$	1	no-equilibrium
$-$	$-$	$+$	> 0	$\gtrless 0$	2	unique
$-$	$-$	$-$	< 0	< 0	3	multiple

Equilibrium is determinate only where $ind(\overline{x}) = 2$. A necessary condition for this case is $\det(\widehat{A}) > 0$. But this is not s sufficient condition as it applies for fixed points with $ind(\overline{x}) = 0$. We can rule out the case where $ind(\overline{x}) = 0$ by requiring that monetary policy also induce $\mathrm{tr}(\widehat{A}) < 0$. To conclude, where fiscal policy is passive, we can make sure that $ind(\overline{x}) = 2$ by requiring that monetary policy should bring about $r_1 r_2 r_3 > 0$ and $r_1 + r_2 + r_3 < 0$.

Note equations (32)-(33). Solving $\frac{\sigma \widetilde{c}^* (\nu + R^*)^2 (\rho+\delta)^2}{\alpha \nu \xi_k} (\alpha - 1) > 0, \rho + (\nu + R^*) \frac{\alpha - 1}{\alpha} + (\rho + \delta)\left(1 + \frac{1}{\nu} R^*\right) < 0$ we obtain that $\alpha < \frac{1}{1 + \frac{\rho}{\nu + R^*} + \frac{\rho + \delta}{\nu}} < 1$.

Consider a passive fiscal stance, i.e. $\gamma > \rho$.

In this regime the eigenvalue $\rho - \gamma$ is negative. Hence, monetary policy must induce that $\widehat{A}$ has only one stable eigenvalue. A necessary condition for this case is $\det(\widehat{A}) < 0$ and to rule out the possibility that monetary policy induces three stable roots we require that $\mathrm{tr}(\widehat{A}) > 0$.

Solving $\frac{\sigma \widetilde{c}^* (\nu + R^*)^2 (\rho+\delta)^2}{\alpha \nu \xi_k} (\alpha - 1) < 0, \rho + (\nu + R^*) \frac{\alpha - 1}{\alpha} + (\rho + \delta)\left(1 + \frac{1}{\nu} R^*\right) > 0$ we obtain that $\alpha > 1$. QED

Proof of Proposition 9 Assume that $\frac{\alpha - 1}{\alpha}\left[\beta + \frac{\tau^*}{1 - \varphi^*} + \frac{\gamma}{\rho \tilde{a}^*}\left(\frac{1}{\nu} R^* - \frac{\tau^* \varphi^*}{1 - \varphi^*}\right)\right] \neq 0$ then the multiple of eigenvalues is non zero which indicates that there is no zero eigenvalue. Assume now that $\frac{\alpha - 1}{\alpha}\left[\beta + \frac{\tau^*}{1 - \varphi^*} + \frac{\gamma}{\rho \tilde{a}^*}\left(\frac{1}{\nu} R^* - \frac{\tau^* \varphi^*}{1 - \varphi^*}\right)\right] = 0$ then either $\alpha = 1$ or $\beta + \frac{\tau^*}{1 - \varphi^*} + \frac{\gamma}{\rho \tilde{a}^*}\left(\frac{1}{\nu} R^* - \frac{\tau^* \varphi^*}{1 - \varphi^*}\right) = 0$. In what follows I show that either policies induce a zero eigenvalue, i.e. that there is a bifurcation at $\alpha = 1$ and given β there is a bifurcation at $\gamma = \rho \tilde{a}^* \frac{\beta + \frac{\tau^*}{1 - \varphi^*}}{\frac{\tau^* \varphi^*}{1 - \varphi^*} - \frac{1}{\nu} R^*}$. Note that implementing $\alpha = 1$ and $\beta + \frac{\tau^*}{1 - \varphi^*} +$

$\frac{\gamma}{\rho\tilde{a}^*}(\frac{1}{\nu}R^* - \frac{\tau^*\varphi^*}{1-\varphi^*}) = 0$ simultaneously, brings about a codimension two bifurcation.

Assume $\alpha = 1$ and $\beta + \frac{\tau^*}{1-\varphi^*} + \frac{\gamma}{\rho\tilde{a}^*}(\frac{1}{\nu}R^* - \frac{\tau^*\varphi^*}{1-\varphi^*}) \neq 0$

Substituting $\alpha = 1$ into equation (56) we obtain that

$$B_{[\alpha=1]} \equiv \begin{bmatrix} 0 & -\frac{\sigma\alpha\widetilde{c}^*(\rho+\delta)}{\nu+R^*}f^* & -\frac{\sigma\widetilde{c}^*}{1+\frac{1}{\nu}R^*}f_k^{*2}\left(\beta + \frac{\tau^*}{1-\varphi^*}\right) & -\gamma\frac{\sigma(\rho+\delta)}{1-\tau^*}\frac{\widetilde{c}^*}{\widetilde{a}^*} \\ 0 & \rho+\delta & \frac{\nu}{\alpha}\frac{f_k^{*2}}{f^*}\left(\beta + \frac{\tau^*}{1-\varphi^*}\right) & \gamma\frac{\nu}{\alpha}\frac{f_k^*}{f^*}\frac{1}{\widetilde{a}^*} \\ B_{3,1} & 0 & B_{3,3} & 0 \\ 0 & B_{4,2} & B_{4,3} & B_{4,4} \end{bmatrix}$$

where $B_{i,j}$ $i,j = 1,..4$ are components of B specified in eq. (56), respectively. Where $\alpha = 1$ the first row is a multiplication of the second row by $-\frac{\sigma\alpha\widetilde{c}^*}{\nu+R^*}f^*$. Consequently $A_{[\alpha=1]}$ is singular.

Assume $\alpha \neq 1$ and $\beta + \frac{\tau^*}{1-\varphi^*} + \frac{\gamma}{\rho\tilde{a}^*}(\frac{1}{\nu}R^* - \frac{\tau^*\varphi^*}{1-\varphi^*}) = 0$.

Substituting $\widetilde{\gamma} \equiv \rho\tilde{a}^*\frac{\beta+\frac{\tau^*}{1-\varphi^*}}{\frac{\tau^*\varphi^*}{1-\varphi^*}-\frac{1}{\nu}R^*}$ into equation (56) we obtain that

$$B_{[\widetilde{\gamma}]} \equiv \begin{bmatrix} 0 & B_{1,2} & B_{1,3} & \psi B_{1,3} \\ 0 & B_{2,2} & B_{2,3} & \psi B_{2,3} \\ B_{3,1} & 0 & B_{3,3} & 0 \\ 0 & B_{4,2} & B_{4,3} & \psi B_{4,3} \end{bmatrix}$$

where $\psi \equiv \frac{\rho}{f_k^*\left[\frac{\tau^*\varphi^*}{1-\varphi^*}-\frac{1}{\nu}R^*\right]}$ is a constant. It is straightforward to notice that the determinant of $B_{[\widetilde{\gamma}]}$ equals zero which implies that $B_{[\widetilde{\gamma}]}$ is singular. Thus, we showed that a monetary-fiscal regime such that $\frac{\alpha-1}{\alpha}\left[\beta+\frac{\tau^*}{1-\varphi^*}+\frac{\gamma}{\rho\tilde{a}^*}(\frac{1}{\nu}R^*-\frac{\tau^*\varphi^*}{1-\varphi^*})\right]=0$ brings about a non hyperbolic equilibrium and this concludes the proof.

Proof of Proposition 10 From Table A.1 it follows that a necessary condition for equilibrium determinacy is $\det(B)>0$. The proof to the proposition is concluded by requiring that the right hand side of equation (57) is positive.

Proof of Proposition 11 Equilibrium is determinate only where $ind(\overline{x})=2$. $B_{[\gamma=0]}$ is block recursive with one positive eigenvalue at the lower right 1×1 submatrix, and so we obtain the dimension of the stable manifold only by examining $\widehat{B_1}$. Observe Table A.2. $\det(\widehat{B_1})>0$ is a necessary condition. Furthermore, we must rule out the case where $ind(\overline{x})=0$ by requiring that policy also induce $\mathrm{tr}(\widehat{B_1})<0$. To conclude, we can make sure that $ind(\overline{x})=2$ by implementing a policy that brings about $r_1r_2r_3>0$ and $r_1+r_2+r_3<0$.

It follows from equation (60) that $r_1r_2r_3>0\Leftrightarrow\frac{\alpha-1}{\alpha}\left[\beta+\frac{\tau^*}{1-\varphi^*}\right]<0$.

So $\det(\widehat{B_1}) > 0$ under two regimes: $\{\alpha > 1$ and $\beta + \frac{\tau^*}{1-\varphi^*} < 0\}$ or $\{\alpha < 1$ and $\beta + \frac{\tau^*}{1-\varphi^*} > 0\}$. Under the first regime, we obtain from equation (61) that $\mathrm{tr}(\widehat{B_1}) > 0$ and we cannot rule out the possibility of No-equilibrium. However, in the second regime where $\alpha < 1$ we can ensure that $\mathrm{tr}(\widehat{B_1}) < 0$ by requiring that $\alpha < \frac{1}{1+\frac{\rho}{\nu+R^*}+\frac{\rho+\delta}{\nu(1-\tau^*)}} < 1$. *QED.*

Proof of Proposition 12 The main idea of this proposition is that perturbations are in the parameter space $\{\gamma\}$. By construction such perturbations do not change the steady state itself however they may change the phase portrait of the steady state. Thus, starting from a determinate equilibrium, as long as γ does not reach its bifurcation point, the phase portrait of the (unchanged) steady state should not be affected by the perturbation. The formal proof of Corollary 4 follows directly from the following Theorems. Specifically, in the terminology of the Theorem 1 and 2 I choose $\overline{y} = \overline{x}$ and $g(y)$ that differs from $f(x)$ up to the perturbation of γ.

Theorem 1 [Hirsch and Smale (1976) Chap.16] *Let $f : W \to E$ be a C^1 vector field and $\overline{x} \in W$ an equilibrium of $\dot{x} = f(x)$ such that*

$Df(\overline{x}) \in L(E)$ is invertible. Then there exists a neighborhood $U \subset W$ of $\overline{x}$ and a neighborhood $\Re \subset \mho(W)$ of f such that for any $g \in \Re$ there is a unique equilibrium $\overline{y} \in U$ of $\dot{y} = g(y)$. Moreover, if E is normed, for any $\epsilon > 0$ we can choose $\Re$ so that $|\overline{y} - \overline{x}| < \epsilon$.

Theorem 2 [Hirsch and Smale (1976) Chap.16] *Suppose that $\overline{x}$ is a hyperbolic equilibrium. In Theorem 1, then, $\Re$ and U can be chosen so that if $g \in \Re$, the unique equilibrium $\overline{y} \in U$ of $\dot{y} = g(y)$ is hyperbolic and has the same index as $\overline{x}$.*

Proof of Proposition 13 By construction the values $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$ ensure that $\overline{x}$ is a hyperbolic equilibrium with index 2. Perturbations to $\overline{\gamma}$ do not change the phase portrait of $\overline{x}$ as long as the multiple of eigenvalues does not change signs. According to proposition 10 $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$ must satisfy $\frac{\overline{\alpha}-1}{\overline{\alpha}}\left[\overline{\beta} + \frac{\tau^*}{1-\varphi^*} + \frac{\overline{\gamma}}{\rho\tilde{a}^*}(\frac{1}{\nu}R^* - \frac{\tau^*\varphi^*}{1-\varphi^*})\right] < 0$ which they do by construction. Note that $\frac{\overline{\alpha}-1}{\overline{\alpha}} > 0$ which implies that $\overline{\beta} + \frac{\tau^*}{1-\varphi^*} + \frac{\overline{\gamma}}{\rho\tilde{a}^*}(\frac{1}{\nu}R^* - \frac{\tau^*\varphi^*}{1-\varphi^*}) < 0$. Also by construction there is some $\varepsilon_2 > 0$ such that $\overline{\beta} + \frac{\tau^*}{1-\varphi^*} = -\frac{\overline{\gamma}}{\rho\tilde{a}^*}(\frac{1}{\nu}R^* - \frac{\tau^*\varphi^*}{1-\varphi^*}) - \varepsilon_2$. We now perturb the system so that $\gamma = \overline{\gamma} + \varepsilon_1$ where ε_1 can be either positive or negative. The equilibrium will have the same index as $\overline{x}$ as long as the perturbed system satisfies

$\overline{\beta} + \frac{\tau^*}{1-\varphi^*} + \frac{\gamma}{\rho\tilde{a}^*}(\frac{1}{\nu}R^* - \frac{\tau^*\varphi^*}{1-\varphi^*}) < 0$

$\Leftrightarrow -\frac{\overline{\gamma}}{\rho\tilde{a}^*}(\frac{1}{\nu}R^* - \frac{\tau^*\varphi^*}{1-\varphi^*}) - \varepsilon_2 + \frac{\overline{\gamma}+\varepsilon_1}{\rho\tilde{a}^*}(\frac{1}{\nu}R^* - \frac{\tau^*\varphi^*}{1-\varphi^*}) < 0$

$\Leftrightarrow -\varepsilon_2 + \frac{\varepsilon_1}{\rho\tilde{a}^*}(\frac{1}{\nu}R^* - \frac{\tau^*\varphi^*}{1-\varphi^*}) < 0$

$\Leftrightarrow \varepsilon_1 > \frac{\rho\tilde{a}^*}{\frac{1}{\nu}R^* - \frac{\tau^*\varphi^*}{1-\varphi^*}}\varepsilon_2$ and any $\varepsilon_1 > 0$ satisfies this condition. $\varepsilon_1 > 0$ is not a necessary condition. However, if we consider left perturbations of γ, the magnitude of admissible perturbations depends on the choice of $\overline{\beta}$. *QED*.

Proof of Proposition 14 According to proposition 10 a stabilizing regime must satisfy $\frac{\alpha-1}{\alpha}\left[\beta + \frac{\tau^*}{1-\varphi^*} + \frac{\gamma}{\rho\tilde{a}^*}(\frac{1}{\nu}R^* - \frac{\tau^*\varphi^*}{1-\varphi^*})\right] < 0$. Note that at the fiscal limit $\frac{1}{\nu}R^* - \frac{\tau^*\varphi^*}{1-\varphi^*} \geq 0$. Also note that a passive fiscal policy implies that $\frac{\gamma}{\rho\tilde{a}^*} > 1$. Thus, the "usual" requirement $\beta + \frac{\tau^*}{1-\varphi^*} > 0$ implies that an active-monetary passive-fiscal regime does not satisfy proposition 10. Suppose that at the fiscal limit the monetary policy switched to a passive regime. In this case the necessary conditions in Proposition 10 are satisfied. However, we can not find sufficient conditions that a passive-monetary passive-fiscal regime is stabilizing.

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