Infinite horizon allocation with consumption-dependent utility

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October 18, 2012

Abstract

We consider an economy in which there is an infinite stream of pies, each of size one, one in every period. For each agent, the per-period utility function, which is defined on that period's consumption, is determined by the previous period's consumption. We describe specifications of this model for which no symmetric, efficient, and monotonic way to allocate pies exists.

Keywords: Behavioral preferences; Dynamic models; Habits.

JEL Codes: D01, D03, D30, D31, D63.

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1 Introduction

Past experience is instrumental in shaping present preferences.[1] In this paper we investigate the implications of this intertemporal dependence on collective resource allocation.

To this end, we consider an infinite horizon economy populated by n long-lived agents who face an infinite stream of pies, in which period t's consumption determines period $(t+1)$'s utility function. There is no saving in the economy, so in each period there is a unit of a homogeneous good that can be divided among the n agents, but cannot be stored. The agents, who care about their discounted sum of utilities, share a common discount factor, $\delta \in (0, 1)$.

The static counterpart of our model is Nash's (1950) *bargaining problem*. Namely, setting $\delta = 0$ in our model boils it down to Nash's model, which has triggered a huge literature, offering a plethora of interesting solutions (see Thomson (1994) for a survey). In our dynamic setting things are considerably more complicated, because today's allocation influences tomorrow's preferences. As a result, finding a satisfactory solution—that is, a satisfactory allocation rule—is a nontrivial task.

We focus on three axioms that the allocation rule is required to satisfy: *efficiency*, *symmetry*, and *monotonicity*. Efficiency is obvious. Symmetry requires all the agents to enjoy the same lifetime payoff if they all start their life with the same periodic utility function, and monotonicity requires none to get hurt if the set of available (lifetime) utility allocations expands. When $\delta = 0$, these axioms determine a unique solution—the *egalitarian bargaining solution*, due to Kalai (1977).[2] In contrast, when δ is positive, an allocation rule satisfying the three axioms may not exist.

[1] The ideas of habit formation and habit persistence, and their economic implications, are well-studied in the literature. Prominent examples include Abel (1990), Boldrin et al. (2001), Campbell and Cochrane (1999), Christiano et al. (2005), Constantinides (1990), Pollak (1970), Ravn and Schmitt-Grohe (2006), and Sundaresan (1989).

[2] See Roemer (1986).

We start by considering the case where $n = 2$, the periodic utility function is logarithmic, and is defined on consumption increments between consecutive periods. Here, there exist a critical value, $\delta^* \in (0, 1)$, such that if the discount factor belongs to $(0, \delta^*)$, then no symmetric, efficient and monotonic allocation rule exists.

Next, we consider the case where the periodic utility function is linear, and the (constant) marginal utility is determined by last period's consumption. For $n = 2$, there does not exist any symmetric and efficient allocation rule, provided that the discount factor is at least one half (this impossibility obtains even without imposing monotonicity). Next, we show that for any number of agents $n \geq 3$, there is a range for the discount factor, $I_n \subset (0, 1)$, such that if $\delta \in I_n$, then a symmetric, efficient and monotonic allocation is impossible. This interval I_n gets closer to zero as n grows, which leads us to investigate whether the impossibility still holds in large societies, in which the agents' discount factor is bounded away from zero. We answer this question positively.

The next section describes the multi-period, Section 3 describes the two specific models and states the results, Section 4 concludes, and the Appendix contains the proofs.

2 The economy

There is a stream of identical pies, each of size one. Time runs as $t = 1, 2, \cdots$ ad infinitum, and there is one pie per period. Pies cannot be stored: in each period, any unconsumed portion of the pie depreciates completely. The economy is populated by n long-lived agents. Given an infinite consumption steam $\{c_t^i\}$, agent i's corresponding total utility is given by the discounted sum $\sum \delta^{t-1} u_t^i(c_t^i)$, where $u_t^i(.)$ is period-t's utility function, $c_t^i \in [0, 1]$ is the portion of the period-t pie this agent consumes, and $\delta \in (0, 1)$ is the discount factor, common to all agents. Our key assumption is:

$$u_{t+1}^i = \psi(c_t^i),$$

where ψ is some function. Namely, each period's utility function is determined by the agent's previous period's consumption.

An *allocation rule* prescribes a feasible allocation of the pie as a function of the agents' utility functions. Formally, with the set of utility functions denoted by $\mathcal{U}$, an allocation rule is a map

$$f\colon \mathcal{U}^n \to \Delta,$$

where $\Delta \equiv \{x \in \mathbb{R}_+^n : \sum x_i \leq 1\}$. For short, we call a specification of utility functions, $(u_1, \cdots, u_n)$, a *state*, and we label it by θ.[3]

Let $V_i(\theta; f)$ denote agent i's value from entering this economy when the state is θ and the allocation rule is f. This value function adheres to the following Bellman equation:

$$V_i(\theta; f) = u_i(f_i(\theta)) + \delta V_i(\psi(f(\theta)); f),$$

where u_i is the i-th component of θ and—slightly abusing notation—$\psi(f(\theta)) \equiv (\psi(f_1(\theta)), \cdots, \psi(f_n(\theta)))$.

Let $\mathcal{S} \subset \mathcal{U}^n$ be the set of those states $\theta = (u_1, \cdots, u_n)$ such that $u_1 = u_2 = \cdots = u_n$. These are the *symmetric* states. An allocation rule f is *symmetric* if $V_1(\theta; f) = V_2(\theta; f) = \cdots = V_n(\theta; f)$ whenever $\theta \in \mathcal{S}$. Note that symmetry does **not** imply that the pie is divided equally among the agents when the state is symmetric

[3] Note that not all $\theta \in \mathcal{U}$ are necessarily *feasible*, in the following sense: it may be the case that there does not exist any $x \in \Delta$ such that $\psi(x) = \theta$. Nothing essential in our analysis would change if, alternatively, we defined the allocation rule to be a map $f\colon \Theta \to \Delta$, where $\Theta \equiv \{\theta \in \mathcal{U}^n : \theta = \psi(x) \text{ for some } x \in \Delta\}$. Defining the domain of the allocation rule as we did, allows it to contain states that cannot be generated by ψ, thus giving us more freedom in the specification of the utility functions in the first period, $t = 1$.

(i.e., $f_1(\theta) = \cdots = f_n(\theta)$ when $\theta \in \mathcal{S}$); the converse, of course, is true.

An allocation rule f is *efficient* if for every other rule g and any state θ, it is **not** the case that $V_i(\theta; g) \geq V_i(\theta; f)$ for all i and there is a strict inequality for some i. Our notion of efficiency may be viewed as strong, because an alternative rule g is not allowed to generate a Pareto improvement, regardless of the state. One may suggest the following more permissive notion of efficiency: to demand that for every g, *if* there is some state θ such that g Pareto improves on f given θ, then there exists another state, θ', such that f Pareto improves on g given θ'. We do not know whether our impossibility results hold under this weaker notion of efficiency. Conceptually, the justification for the stronger notion we adopt stems from sequential rationality. Think of the allocation as determined by a planner; if the planner is present in the economy at all dates, he can readjust the allocation at any time—nothing constraints him to decide in advance on future allocations. Since he can revise allocations given any state θ, it better be the case that efficiency obtains at every θ.

Given $u, v \in \mathcal{U}$, write $u \geq v$ if $u(x) \geq v(x)$ for all $x \in [0, 1]$. Given two states $\theta = (u_1, \cdots, u_n)$ and $\theta' = (u'_1, \cdots, u'_n)$, write $\theta \geq \theta'$ if $u_i \geq u'_i$ for all i. An allocation rule f is *monotonic* if $V_i(\theta; f) \geq V_i(\theta'; f)$ for all i, provided that $\theta \geq \theta'$. This is an analog of Kalai's (1977) monotonicity axiom from bargaining theory, which requires expansion of options in the utility space not to hurt any agent.

3 Results

Below we specify two instances of the general setting that has just been described in Section 2 above—the *logarithmic model* and the *linear model*. In either case, symmetric, efficient, and monotonic allocation is impossible for some set of model parameters. The idea behind the proofs is to start with a symmetric state and a symmetric allocation, and show that by "breaking down" the allocation's symmetry one can generate a Pareto improvement.

In either model tomorrow's utility function is decreasing in today's consumption, so if we transfer some resources from agent i to agent j, we increase j's current periodic payoff and increase i's *utility function* in the next period. Depending on the specific parameters at hand (the periodic utility function, the discount factor, and the number of agents) various such improving-deviations can be constructed at the expense of symmetry.[4]

3.1 The logarithmic model

We start by considering the following preferences, which are common in the literature (see, e.g., Christiano et al. (2005) for a similar specification). Given period $(t-1)$'s consumption, c_{t-1}, the period-t utility, defined on c_t, takes the form

$$u_{c_{t-1}}(c_t) = \log(1 + \max\{0, c_t - c_{t-1}\}).$$

Call the resulting model the *logarithmic model.*

Theorem 1. *Assume the logarithmic model and suppose that $n = 2$. There exists a $\delta^* \in (0,1)$ such that if $\delta \in (0, \delta^*)$ then there does not exist a symmetric, efficient, and monotonic allocation rule.*

3.2 The linear model

Here we consider the case where $u_t^i(c_t^i) = \alpha_t^i \times c_t^i$ and the rule ψ maps consumption levels to marginal utilities. Specifically, we consider the linear transition rule

$$\alpha_{t+1}^i = 1 - c_t^i. \tag{1}$$

[4]Monotonicity is required in the proof of three out of our four results. As the informal explanation from above indicates, we believe that it is not essential; however, we do not know whether it can be dispensed with.

The idea behind (1) is that if an agent eats "a lot" today, then he will need to eat "a whole lot" tomorrow in order to arrive at the same utility level. Conversely, present hunger makes the agent more appreciative of food in the next period, in the sense of having higher marginal utility. Capturing this pattern is not possible within the confines of the logarithmic model, in which an increment in today's consumption increases tomorrow's marginal utility.

Call the resulting model the *linear model*.

Theorem 2. *Assume the linear model with $n = 2$. If $\delta \geq \frac{1}{2}$, then there does not exist any symmetric and efficient allocation rule.*

Note that monotonicity is not invoked in this theorem. In the subsequent theorems—Theorems 3 and 4—we do impose monotonicity.

Theorem 3. *Assume the linear model. For every $n \geq 3$ there exists a non-degenerate interval, $I_n \subset (0, 1)$, such that if the agents' common discount factor lies in I_n, then there does not exist a symmetric, efficient, and monotonic allocation rule.*

The discount factor around which the interval I_n is constructed takes the form $\frac{1}{n-1}$. Thus, the range of discount factors for which the result applies shrinks to zero as n grows. Does impossibility obtain when the agents are highly patient, or at least do not become perfectly impatient as n grows? We do not know whether an analog of Theorem 3 holds for large societies consisting of agents with discount factors arbitrarily close to one, but we can nonetheless show that the discount factor for which impossibility obtains can be bounded away from zero.

Theorem 4. *For every $\epsilon > 0$ there exist an $n = n(\epsilon)$ and a non-degenerate interval $J = J(\epsilon) \subset (0, 1)$ such that the following is true: (1) There does not exist a symmetric, efficient, and monotonic allocation rule in the linear model with $N \geq n$ agents whose common discount factor lies in J, and (2) $dist(J(\epsilon), \frac{1}{e}) \to 0$ as $\epsilon \to 0$, where $dist(A, a)$ is the Hausdorff distance between the set A and the point a.*

Theorem 4 is proved by showing that for a sufficiently large n, there does not exist a symmetric, efficient, and monotonic allocation rule if the discount factor takes the form $\delta = (\frac{n}{n+1})^{\frac{1}{n-1}}$.

4 Closing comments

We have introduced an extension of Nash's (1950) bargaining model that expresses the dependence of current preferences on past experience. Within this model, we have shown that allocating goods in symmetric, efficient, and monotonic way—which in Nash's model is equivalent to egalitarianism—may be impossible.

Our model is restrictive in (at least) two ways. First, the allocation rule, by definition, it recursive: only information from the previous period is needed in order to decide on today's allocation. Second, allocations are not allowed to be stochastic. It remains to be seen whether stochastic allocations and/or rules that condition on long histories can solve the nonexistence problem in those cases where it otherwise arises.

Acknowledgments: We thank Alan Miller for his helpful comments.

Appendix: Proofs

Proof of Theorem 1

Assume by contradiction that an efficient and symmetric allocation rule exists. Let $W_i(c_1, c_2)$ be agent i's value in the state where last period's consumptions are given by (c_1, c_2). For the case where $c_1 = c_2 \equiv c$, denote by $V(c) \equiv W_1(c, c) = W_2(c, c)$ the common value. The Bellman equation for V is

$$V(c) = \log(1 + \max\{0, x_i - c\}) + \delta W_i(x_1, x_2),$$

where x_i is the pie share of agent i. Consider $c = \frac{1}{2}$. Let $x = (x_1, x_2)$ be the allocation given this state. Wlog, suppose that $x_1 \leq \frac{1}{2}$. Then

$$V(\frac{1}{2}) = \delta W_1(x_1, x_2).$$

Moreover,

$$V(\frac{1}{2}) = \delta W_1(x_1, x_2) \leq \delta W_1(0, 0) = \delta V(0),$$

because the inequality follows from monotonicity. Therefore,

$$V(\frac{1}{2}) \leq \delta V(0) \tag{2}$$

Now, consider the following policy, which has the following prescriptions when both agents have past period consumption equal to $\frac{1}{2}$: agent 1 is allocated the entire pie today and then every two periods, whereas agent 2 receives nothing today, the entire pie tomorrow, and then the entire pie every two periods (i.e., the agents alternate roles in having the whole pie). The value for agent 1 is $\log\frac{3}{2} + \delta^2 \frac{\log 2}{1-\delta^2}$ and for agent 2 it is $\delta \frac{\log 2}{1-\delta^2}$. It is easy to verify that the value for agent 1 is higher than agent 2's value (for all $\delta > 0$). By efficiency, it must be that $V(\frac{1}{2})$ is at least as large as agent 2's value from the alternative policy. Namely, $\delta \frac{\log 2}{1-\delta^2} \leq V(\frac{1}{2})$. Invoking (2) we obtain

$$\delta \frac{\log 2}{1-\delta^2} \leq \delta V(0),$$

or

$$\frac{\log 2}{1-\delta^2} \leq V(0).$$

Denoting by (z_1, z_2) the allocation at $c = 0$, we have

$$V(0) = \log(1 + z_i) + \delta W_i(z_1, z_2) \geq \frac{\log 2}{1-\delta^2}.$$

Because $W_i \leq \frac{\log 2}{1-\delta}$ and $z_i \leq \frac{1}{2}$ for at least one i, it follows that $\log\frac{3}{2} + \delta \frac{\log 2}{1-\delta} \geq \frac{\log 2}{1-\delta^2}$, which is violated for all δ's below a certain threshold. $\square$

Proof of Theorem 2

To prove this theorem, we start by formulating and solving an auxiliary problem. Consider the liner model with $n = 1$. Namely, a single agent is facing an infinite sequence of size-one pies, and he needs to decide how much to consume in every period. His periodic utility function, defined on that period's consumption, c, is $u(c) = \alpha \times c$, where $1 - \alpha$ is the consumption in the previous period. The value function for this agent's problem is:

$$V(\alpha) = \max_{x \in [0,1]} \ \alpha x + \delta V(1-x). \tag{3}$$

Define the function V^* as follows:

$$V^*(\alpha) = \max\{\frac{\delta}{1-\delta^2}, \alpha + \frac{\delta^2}{1-\delta^2}\}.$$

Lemma 1. *V^* is the unique solution to (3).*

Proof. It is easy to see that the RHS of (3) satisfies Blackwell's sufficient conditions for contraction, hence a unique solution exists. It is enough to verify that V^* is a solution.

Plugging V^* into the RHS of (3) we obtain that it is

$$\max_{x \in [0,1]} \ \alpha x + \delta \times \{\max[\frac{\delta}{1-\delta^2}, 1 - x + \frac{\delta^2}{1-\delta^2}]\}. \tag{4}$$

We will now solve the maximization program (4). To this end, we consider three cases.

Case 1: $\alpha \leq \frac{\delta}{1+\delta}$. Let x^* be a maximizer of the above equation. If $\max[\frac{\delta}{1-\delta^2}, 1 - x^* + \frac{\delta^2}{1-\delta^2}] = \frac{\delta}{1-\delta^2}$, then the corresponding values is $\alpha x^* + \frac{\delta^2}{1-\delta^2} \leq \alpha + \frac{\delta^2}{1-\delta^2} < \frac{\delta}{1-\delta^2}$. However, note that by choosing $x = 0$, the agent obtains the value $\frac{\delta}{1-\delta^2}$. Therefore, the maximizer x^* must be such that $\frac{\delta}{1-\delta^2} < 1 - x^* + \frac{\delta^2}{1-\delta^2}$ and the corresponding value is $\alpha x^* + \delta\{1 - x^* + \frac{\delta^2}{1-\delta^2}\}$. Note that the derivative with respect to x^* is $\alpha - \delta < \frac{\delta}{1+\delta} - \delta < 0$, hence $x^* = 0$. Therefore, the maximization's value is $\frac{\delta}{1-\delta^2}$.

Case 2: $\frac{\delta}{1+\delta} < \alpha < \delta$. Again, let x^* be a maximizer. If $\max[\frac{\delta}{1-\delta^2}, 1 - x^* + \frac{\delta^2}{1-\delta^2}] = 1 - x^* + \frac{\delta^2}{1-\delta^2}$, then the value is $\alpha x^* + \delta\{1 - x^* + \frac{\delta^2}{1-\delta^2}\}$. Note that the derivative with respect to x^* is $\alpha - \delta < \delta - \delta = 0$, so $x^* = 0$ and the value is $\frac{\delta}{1-\delta^2}$. However, by setting $x = 1$ the agent can guarantee to himself $\alpha + \frac{\delta^2}{1-\delta^2} > \frac{\delta}{1-\delta^2}$. Therefore $\frac{\delta}{1-\delta^2} > 1 - x^* + \frac{\delta^2}{1-\delta^2}$, hence the objective assumes the form $\alpha x + \frac{\delta^2}{1-\delta^2}$, which is maximized at $x = x^* = 1$, giving rise to the value $\alpha + \frac{\delta^2}{1-\delta^2}$.

Case 3: $\alpha \geq \delta$. If $\frac{\delta}{1-\delta^2} > 1 - x^* + \frac{\delta^2}{1-\delta^2}$ then the maximum is at $x^* = 1$ and the value is $\alpha + \frac{\delta^2}{1-\delta^2}$. Otherwise, the objective assumes the form $\alpha x^* + \delta\{1 - x^* + \frac{\delta^2}{1-\delta^2}\}$, which is nondecreasing in x^*. Since the objective is continuous in x, the value is $\alpha + \frac{\delta^2}{1-\delta^2}$.

Let $v = v(\alpha)$ denote the value of the maximization (4). We have just shown that:

$$v(\alpha) = \begin{cases} \frac{\delta}{1-\delta^2} & \text{if } \alpha \leq \frac{\delta}{1+\delta} \\ \alpha + \frac{\delta^2}{1-\delta^2} & \text{otherwise} \end{cases}$$

That is, $v = V^*$. □

Equipped with the solution to the single agent case, we can turn back to the 2-agent model. Theorem 2 is now proved easily via the following lemmas.

Lemma 2. *Suppose that $\delta \geq \frac{1}{2}$. Let f be a symmetric allocation rule for the linear 2-agent model. Let $\alpha_1 = \alpha_2 = \frac{1}{2}$. Then the corresponding value for each agent is at most $\frac{\delta}{1-\delta^2}$.*

Proof. Assume by contradiction that the value (which is common to both agents) is strictly above $\frac{\delta}{1-\delta^2}$. Wlog, suppose that the share of agent 1 from today's pie, say x, is weakly smaller than one half. We therefore have $\frac{x}{2} + \delta W > \frac{\delta}{1-\delta^2}$, where W is agent 1's continuation value. Since $\delta \geq \frac{1}{2}$, it follows that $\delta(x + W) > \frac{\delta}{1-\delta^2}$, and since $x \leq \frac{1}{2}$ it follows that $W > \frac{1}{1-\delta^2} - \frac{1}{2} = \frac{1+\delta^2}{2(1-\delta^2)}$.

On the other hand, $W \leq V^*$, where V^* is the value function from the single agent problem, which is weakly increasing. Therefore $W \leq V^*(x) \leq V^*(\frac{1}{2}) = \frac{1}{2} + \frac{\delta^2}{1-\delta^2} = \frac{1+\delta^2}{2(1-\delta^2)}$. Combining the bounds on W we obtain $\frac{1+\delta^2}{2(1-\delta^2)} < \frac{1+\delta^2}{2(1-\delta^2)}$, a contradiction. □

Lemma 3. *Consider the linear 2-agent model and suppose that $\alpha_1 = \alpha_2 = \frac{1}{2}$. Then there is an allocation that gives one agent the value $\frac{\delta}{1-\delta^2}$ and gives the other agent a value which is strictly larger that $\frac{\delta}{1-\delta^2}$.*

Proof. Consider the following allocation: give the entire pie to agent 1 today, give the entire pie to agent 2 tomorrow, and let them "alternate roles" forever onwards: each receives the entire pie every other period. Clearly, the value for agent 2 is exactly $\frac{\delta}{1-\delta^2}$. For agent 1 it is $\frac{1}{2} + \frac{\delta^2}{1-\delta^2} > \frac{\delta}{1-\delta^2}$. □

Proofs of Theorems 3 and 4

We start with a lemma that will be handy in the proof of either theorem.

Consider a symmetric, monotonic, and efficient rule, f. By symmetry, we can define the following function:

$$U(\alpha) \equiv V_i((\alpha, \cdots, \alpha); f).$$

Lemma 4. $U(1) < \frac{\frac{1}{n}}{1-\delta}$.

Proof. Let $x_i \equiv f_i((1, \cdots, 1))$. We have

$$U(1) = x_i + \delta V_i((1 - x_1, \cdots, 1 - x_n); f) \leq x_i + \delta U(1),$$

where the inequality follows from monotonicity. Therefore,

$$U(1) \leq \frac{x_i}{1-\delta}.$$

Since the above inequality holds for all i,

$$U(1) \leq \frac{\frac{1}{n}}{1-\delta}. \tag{5}$$

Now, assume by contradiction that (5) holds as equality. Then $x_i = \frac{1}{n}$ for all i. Therefore $U(1) = \frac{1}{n} + \delta U(\frac{n-1}{n})$, so $U(\frac{n-1}{n}) = U(1) = \frac{\frac{1}{n}}{1-\delta}$. Let $y_i \equiv f_i(\frac{n-1}{n} \cdot \mathbf{1})$. We have

$$\frac{\frac{1}{n}}{1-\delta} = U(\frac{n-1}{n}) \leq \frac{n-1}{n} y_i + \delta U(1) = \frac{n-1}{n} y_i + \delta \frac{\frac{1}{n}}{1-\delta},$$

and therefore $y_i \geq \frac{1}{n-1}$ for all i—a contradiction.

□

Proof of Theorem 3: We have just established that in the situation where all marginal utilities are one, the value for an agent is strictly smaller than $\frac{\frac{1}{n}}{1-\delta}$. We will construct a rule that gives each agent at least $\frac{\frac{1}{n}}{1-\delta}$ in this state, provided that $\delta = \frac{1}{n-1}$. It will then follow from continuity (of the value functions) that such improvements also exist when the discount factor is sufficiently close to $\frac{1}{n-1}$.

Let g be an allocation rule that operates as follows when all marginal utilities are one. It splits the pie equally among agents $\{1, 2, \cdots, n-1\}$ (each receives $\frac{1}{n-1}$), and in the next period it gives the entire pie to agent n. The rule is defined in the same way on all future 2-period blocks; i.e., two periods from now the pie will be split equally among agents $\{1, 2, \cdots, n-1\}$, a period later agent n will receive the entire pie, and so on.

Under this alternative policy each agent $i < n$ receives the value $\frac{\frac{1}{n-1}}{1-\delta^2}$ and agent n's value is $\frac{\delta}{1-\delta^2}$. Note that for $\delta = \frac{1}{n-1}$ the two values coincide. Moreover, this common value is $\frac{n-1}{n(n-2)}$, which also equals $\frac{\frac{1}{n}}{1-\delta}$ when $\delta = \frac{1}{n-1}$. □

Proof of Theorem 4: Consider the situation where all marginal utilities are one. By Lemma 4, it is enough to show that there exists an alternative policy, call it g, under which each agent's value is at least as large as $\frac{\frac{1}{n}}{1-\delta}$.

Consider then the following policy. In the current period, we give the entire pie to agent 1; in the next period we give the entire pie to agent 2, a period after that to agent 3, and so one; n periods from now we give it again to agent 1 and this process is repeated indefinitely. Under this scheme, once an agent receives the pie, he needs to wait n periods until the next time he receives it, hence the effective discount factor is δ^n. When the first cycle starts, the agent with the lowest value is the last one, agent n, hence his value is $\frac{\delta^{n-1}}{1-\delta^n}$. Therefore, it is enough to establish

$$\frac{\delta^{n-1}}{1-\delta^n} \geq \frac{\frac{1}{n}}{1-\delta}.$$

This simplifies to $\frac{1-\delta}{1-\delta^n} \geq \frac{1}{n}\delta^{1-n} \Rightarrow n(1-\delta) \geq \delta^{1-n} - \delta \Rightarrow n-(n-1)\delta \geq \delta^{1-n}$. Multiplying both sides by δ^{n-1} gives $n\delta^{n-1} - (n-1)\delta^n \geq 1$, or $n\delta^{n-1} \geq 1 + (n-1)\delta^n$. Therefore

$$(n-1)\delta^{n-1} + \delta^{n-1} \geq 1 + (n-1)\delta^n,$$

or

$$(n-1)\delta^{n-1}(1-\delta) \geq 1 - \delta^{n-1}.$$

So, it is enough to establish that the following holds for all sufficiently large n's:

$$\delta^{n-1}(1-\delta) \geq 1 - \delta^{n-1},$$

It is therefore enough to establish that the following holds for all sufficiently large n's:

$$\delta^{n-1}(2-\delta) \geq 1. \tag{6}$$

Plugging $\delta = (\frac{n}{n+1})^{\frac{1}{n-1}}$ into (6) we obtain

$$\frac{n}{n+1}[2 - (\frac{n}{n+1})^{\frac{1}{n-1}}] \geq 1. \tag{7}$$

The LHS of (7) converges to $2 - e^{-1} > 1$ as $n \to \infty$. $\square$

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