THE ASYMPTOTIC CORE, NUCLEOLUS AND SHAPLEY VALUE OF SMOOTH MARKET GAMES WITH SYMMETRIC LARGE PLAYERS

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Abstract. We study the asymptotic nucleolus of a smooth and symmetric oligopoly with an atomless sector. We show that under appropriate assumptions, the asymptotic nucleolus of the TU market game coincides with the unique TU competitive payoff distribution x^*. This equivalence results from nucleolus of a finite game belonging to its core and the Aumann Core Equivalence, which holds for this economy due to the cut-throat competition among the identical large players. A comparison with the Shapley value yields that in some cases, the asymptotic Shapley value is more favorable for the large traders than the asymptotic nucleolus x^*. This may be interpreted by the 'fairness property' of Shapley Value which does not reflect the intense competition among the large traders, accounting for the relative importance of their marginal contribution. J.E.L. *Classification numbers.* C71, D40, D43.

1. Introduction

We examine a mixed differentiable game, proposed by Aumann [1], with a few identical big (atomic) players and a continuum of small ones. Whether the big players get a better allocation in a core as compared to a competitive one in a mixed game has been analysed in general by Shitovitz [11], and in this work we continue investigating the "power" of big players under alternative solution concepts.[1] There are several types of players. Each type of player has a "corner endowment". The worth of a coalition is the maximal output they can produce using their aggregate endowments and all the technologies available to them. Our main result is that the asymptotic core (the limit of the cores

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[1]See [5] for a comprehensive survey of the known results for the core of mixed games.

of the finite games converging to the initial one, i.e., games in the *admissible sequence*) of the game with identical big players coincides with the competitive (transferable utility competitive equilibrium, or TUCE) payoff. As a consequence, the asymptotic nucleolus (defined for finite games by [9]) is the TUCE payoff distribution as well. This is in contrast with the monopoly case, studied in [3], where nucleolus is strictly better for the monopoly than the TUCE (hence nucleolus capturing the additional "power" enjoyed by the monopoly). Moreover, we conjecture that each of the identical atoms gets a higher payoff under asymptotic Shapley value than in the asymptotic core.

2. The Model

2.1. **Agents.** Let $T(= T_0 \cup T_1)$ be the set of players, Σ be the set of all possible coalitions of the players and λ be the "population measure" so that (T, Σ, λ) is a measure space, where λ is a $\sigma-$additive positive measure on Σ. The set of players is partitioned into T_1, big players (the atoms of λ) and T_0, the small players, i.e., $\lambda = \lambda_0 + \lambda_1$, where λ_0 is non-atomic (with support T_0), and λ_1 has finite support T_1 and coincides with the counting measure.

Let $\mu = (\mu_1, .., \mu_M)$ be a vector of countably additive non-trivial measures absolutely continuous with respect to λ.

Let $f : \mathbb{R}^M_+ \to \mathbb{R}$ be a non-decreasing concave and homogeneous of degree one, zero on the boundaries, continuously differentiable in a neighbourhood of $\mu(T)$. Let $p = \nabla f(\mu(T))$. Assume $p \in \mathbb{R}^M_+$, $p \neq 0$.

2.2. **The induced game.** For all $S \in \Sigma$, $V(S) = f(\mu(S))$.

Remark 1. Define $x^*(S) = p\mu(S)$ for all $S \in \Sigma$. Since f is homogeneous of degree one, by Euler theorem, $p\mu(T) = f(\mu(T))$ and by concavity of f for all $z \in \mathbb{R}^M_+$, $f(z) \leq f(\mu(T)) + p(z - \mu(T)) = pz$. Thus for all $S \in \Sigma$, $x^*(S) = p\mu(S) \geq f(\mu(S)) = V(S)$. It follows that x^* is in the core of V.

2.3. **Players' types.** There are L types of players, so T is partitioned by $\{A_l\}_l$: $\cup_l A_l = T$, for any $l \neq l'$, $A_l \cap A_{l'} = \emptyset$. $\frac{d\mu_m}{d\lambda}$ for all $m \in \{1, \ldots, M\}$ is constant for any l on $T_0 \cap A_l$ and μ is constant for any l on the atoms of $T_1 \cap A_l$.

Let $r \geq 2$ be the greatest common divisor of the numbers of big players of different types: $\lambda(T_1 \cap A_1), \ldots, \lambda(T_1 \cap A_L)$. Let $S_r = \frac{1}{r}T$ be a coalition of players such that for every $S' \subset S_r$, there is $S \subset T$, containing players of the same types as S' such that $\lambda(S) = \lambda(S')r$, so that T is an r-replica of S_r.

2.4. **The admissible sequence of finite games.**

Definition 1. An *admissible sequence* of finite games is defined by an increasing sequence, $(\pi_n)_{n=1}^{\infty}$, of finite subfields of Σ, with the corresponding set of all atoms Π_n of π_n, such that

(1) $\pi_n \subset \pi_{n+1}$ for every $n = 1, 2 \ldots$;
(2) every subset of T_1 is in π_1;
(3) for every l, $T_0 \cap A_l \in \pi_1$;
(4) for any n, any two "small" players of the same type $i_n, j_n \in \Pi_n$, $i_n, j_n \subset T_0 \cap A_l$, are of the same size $\lambda(i_n) = \lambda(j_n)$;
(5) r is also a divisor of $\frac{\lambda(A_l \cap T_0)}{\lambda(i_n)}$ $\forall i_n \in \Pi_n$, $i_n \subset T_0 \cap A_l$, i.e., for every n there is a natural number k such that $\lambda(A_l \cap T_0) = kr\lambda(i_n)$;
(6) and $\cup_{n=1}^{\infty} \pi_n$ generates Σ.

Remark 2. So, Π_n can be thought of as the set of all players of the finite game, with the set of all its subsets (coalitions) being identified with π_n, as in [3, p.191].

Denote the restriction of the game V to π_n by V_n.

Remark 3. By def. 1.(4)-(5), the definition of the core and [2] for any n any two "small" players of the same type $i_n, j_n \in \Pi_n$, $i_n, j_n \subset T_0 \cap A_l$ and any two big players of the same type have the "equal treatment" property. The same holds for the nucleolus since there any two exchangeable players get the same allocation.

3. Results

3.1. **Asymptotic core of the induced game.**

Theorem 1. *Let x_n be in the core of V_n. Then* $\lim_{n\to\infty} x_n(S) = p\mu(S)(= x^*(S))$ *for every* $S \subset T_0$, $S \in \pi_1$.

Proof. By rem. 1 the core of any game in the admissible sequence is non-empty. Since S is in π_1, it is a finite union of some disjoint coalitions $i \in \pi_1$ (or players in Π_1). S being a subset of T_0 implies that all such i are subsets of T_0 too. Since the admissible sequence is increasing, $S \in \pi_n$ for any n. Let $B_n \subset \Pi_n$ be the finite set of disjoint $i_n \subset T_0$ such that their union is S.

For any game V_n and any player $i_n \in B_n$, since x_n is a core allocation,

$$x_n(\tfrac{1}{r}T) + x_n(i_n) = x_n(\tfrac{1}{r}T \cup i_n) \geq f(\mu(\tfrac{1}{r}T) + \mu(i_n)),$$
$$x_n(\tfrac{1}{r}T) - x_n(i_n) = x_n(\tfrac{1}{r}T \backslash i_n) \geq f(\mu(\tfrac{1}{r}T) - \mu(i_n))$$

Since x_n is additive (being a measure) and by rem. 3 $x_n(\frac{1}{r}T) = \frac{1}{r}x_n(T)$, and by homogeneity of degree one of f, $f(\mu(\frac{1}{r}T)) = \frac{1}{r}f(\mu(T))$. As $x_n(T) = V_n(T) = f(\mu(T))$, it follows that $x_n(\frac{1}{r}T) = f(\mu(\frac{1}{r}T))$. Combining with the inequalities above, we get

$$f(\mu(\tfrac{1}{r}T) + \mu(i_n)) - f(\mu(\tfrac{1}{r}T)) \leq x_n(i_n) \leq f(\mu(\tfrac{1}{r}T)) - f(\mu(\tfrac{1}{r}T) - \mu(i_n))$$

By the homogeneity of f, $\nabla f(\mu(T)) = \nabla f\left(\frac{1}{r}\mu(T)\right)$ and since f is (continuously) differentiable in the neighbourhood of $\mu(T)$, it is so in an ε-neighbourhood of $\frac{1}{r}\mu(T) = \mu(\frac{1}{r}T)$. For any such ε there is N_0 sufficiently large, such that for all $n > N_0$ one can assure $\mu(i_n) < \varepsilon$ for any $i_n \in B_n$, so for $n > N_0$, by concavity of f, the last inequality can be re-written as

$$\nabla f\big(\mu(\tfrac{1}{r}T) + \mu(i_n)\big)\mu(i_n) \le x_n(i_n) \le \nabla f\big(\mu(\tfrac{1}{r}T) - \mu(i_n)\big)\mu(i_n)$$

By continuity of ∇f in the ε-neighbourhood of $\mu(\frac{1}{r}T)$, there is $\bar{\delta} \in \mathbb{R}^M_{++}$ such that for any $0 < \delta < \bar{\delta}$ there is $N_1 \ge N_0$ such that for any $n > N_1$, $\nabla f\big(\mu(\frac{1}{r}T) + \mu(i_n)\big) \ge \nabla f(\mu(\frac{1}{r}T)) - \delta = p - \delta$. Similarly, there is $N_2 > N_0$ such that $\forall n > N_2$, $\nabla f\big(\mu(\frac{1}{r}T) - \mu(i_n)\big) \le p + \delta$. So, for all $n > \max\{N_1, N_2\}$ the inequality becomes

$$\big(p - \delta\big)\mu(i_n) \le x_n(i_n) \le \big(p + \delta\big)\mu(i_n)$$

Summing over all $i_n \in B_n$,

$$\big(p - \delta\big)\mu(S) \le x_n(S) \le \big(p + \delta\big)\mu(S)$$

Taking the limit as $\delta \to 0$, we get the result. □

3.2. **Asymptotic nucleolus of the induced game.**

Corollary 1. *Let $\mathcal{N}_n$ be a nucleolus of n-th game in the admissible sequence. Then* $\lim_{n\to\infty} \mathcal{N}_n(S) = p\mu(S)$ *for every $S \subset T_0$, $S \in \pi_1$.*

Proof. By [9, thm. 4] nucleolus belongs to every non-empty core of a finite game. The core of every game in the admissible sequence is non-empty by remark 1. The result follows from thm. 1. □

3.3. **The market game.** Now assume that players of type l have the same concave homogeneous-of-degree-one technology $F_l \colon \mathbb{R}^M_+ \to \mathbb{R}$, strictly increasing, differentiable on $\mathbb{R}^M_{++}$, zero on the boundaries; and the same endowment, $\omega(t) = \omega_l \in \mathbb{R}^M_+$, $\omega_l \ne 0$ for $t \in A_l$. Each type has a *corner endowment*: for every $l \in \{1, \dots, L\}$ there is an input $m_l \in \{1, \dots, M\}$ such that any player of type l has this input: $\omega_l^{m_l} > 0$, but none of the other types has it: $\forall l' \ne l\ \omega_{l'}^{m_l} = 0$. All the big players of the same type have the same measure. So, for any S let $\mu(S) = \sum_{l=1}^{L} \omega_l \lambda(S \cap A_l)$. Then μ_m is a countably additive non-trivial measure, absolutely continuous with respect to λ for any $m \in \{1, \dots, M\}$.

As in [10], [1], define the market game as follows.

Definition 2. For any $S \in \Sigma$, let an $S-$*feasible* allocation be a profile $x = (x_1, \dots, x_M)$, such that for any $m \in \{1, \dots, M\}$

$$\int_S x_m(t) d\lambda(t) \le \mu_m(S) \tag{1}$$

$$v(S) = \sup\Big\{\sum_l \int_{S\cap A_l} F_l(x(t)) d\lambda(t)\Big\} \tag{2}$$

where the supremum is taken over all $S-$feasible allocations x.

Notation 3.1. Denote by $\bar{\Sigma}$ the set of coalitions that contain all types: $S \in \bar{\Sigma} \Rightarrow \lambda(A_l \cap S) > 0 \ \forall l$.

A collection of integrals $\left(\int_A x_m(t) d\lambda(t)\right)_{m=1}^{M}$, $A \subset T$, $x_m(t) \in \mathbb{R}_+$ will be denoted simply by $\int_A x(t) d\lambda(t)$ for $x(t) \in \mathbb{R}_+^M$.

3.4. **The induced form representation of the market game.**

Theorem 2. *There exists $f\colon \mathbb{R}_+^M \to \mathbb{R}$ such that $v = f \circ \mu$, where f is a non-decreasing concave homogeneous of degree one function, zero on the boundaries, and continuously differentiable in the interior.*

Proof. First, $v(S) = 0$ for any coalition that does not contain a positive fraction of at least one type, since then the integral over all F_l is zero. Next, $v(S)$ is bounded for any coalition $S \in \bar{\Sigma}$ (that contains all types): by concavity, for any type l

$$\int_{S \cap A_l} F_l(x(t)) d\lambda(t) \le \lambda(S \cap A_l) F_l\Big(\frac{1}{\lambda(S \cap A_l)} \int_{S \cap A_l} x(t) d\lambda(t)\Big) \tag{3}$$

so, the boundedness follows by $S-$feasibility of allocation x (def. 2) (implying the last integral is finite). Second, the supremum is attained: by inequality 3, for any S the maximization problem reduces to the problem of optimal allocation of inputs across L technologies, i.e., finding $(X_l)_{l=1}^{L} \in \mathbb{R}_+^{ML}$ that maximizes continuous function $(X_l)_l \mapsto \sum_l \lambda(S \cap A_l) F_l(X_l)$ on a compact support $\sum_l \lambda(S \cap A_l) X_l \le \int_S \omega(t) d\lambda(t)$. By homogeneity of F_l, this problem is equivalent to

$$\max_{x_l} \{\sum_l F_l(\lambda(S \cap A_l) X_l) \colon \sum_l \lambda(S \cap A_l) X_l \le z\}$$
$$= \max_{y_l} \{\sum_l F_l(y_l) \colon \sum_l y_l \le z\}$$

Define $f(z) = \max_{y_l}\{\sum_l F_l(y_l) \colon \sum_l y_l \le z\}$. So, $f(\mu(S)) = v(S)$. The required properties of f follow from those of F_l. $\square$

3.5. **Uniqueness of the asymptotic core for symmetric oligopoly.**

Corollary 2. *If every big player $t \in T_1$ is of the same type, then the asymptotic core is x^*, since the "equal treatment" applies to big players, each of whom gets $\frac{p(\mu(T) - \mu(T_0))}{|T_1|} = p\mu(\{t\})$.*

Corollary 3. *If every big player in T_1 is of the same type, then the asymptotic nucleolus is also x^*.*

3.6. **A transferable utility competitive equilibrium.**

Definition 3. A *transferable utility competitive equilibrium* (TUCE) of the economy is a T-feasible allocation z and a price $p \in \mathbb{R}_+^M$, $p \ne 0$ such that for every $t \in A_l$ and $\forall l$, $z^*(t)$ solves $\max_g U_l(g, p\omega(t) - pg)$, where $U_l(g, y) : \mathbb{R}_+^M \times \mathbb{R} \to \mathbb{R} : U_l(g, y) \equiv F_l(g) + y$.

For a TUCE price p define a TUCE *payoff distribution* to be $u^*(S) = \sum_l \int_{A_l \cap S} U_l(z^*(t), p\omega(t) - pz^*(t)) d\lambda(t)$.

3.7. The asymptotic core is a TUCE allocation.

Lemma 1 (based on Einy, Moreno, and Shitovitz [3]). *In a symmetric TUCE (or if F_l are strictly quasiconcave) $u^* = x^*$.*

Proof. First, the TUCE price has to be strictly positive for all inputs, $p \in \mathbb{R}^M_{++}$, since if the price of at least one commodity is zero, at least one of the types will have an unbounded demand which can not be covered by the bounded aggregate endowment of that input. Therefore, since $z^*(t)$ is optimal, it is either zero or strictly positive: $z^*(t) \in \mathbb{R}^M_{++}$, so $\nabla F_l(z^*(t))$ is well defined and equal to p. Since F_l is homogeneous of degree one, for every l, and every $x \in \mathbb{R}^M_{++}$, $F_l(x) = \nabla F_l(x)x$. If $z^*(t) \in \mathbb{R}^M_{++}$, then the equilibrium payoff of this player is $pz^*(t) + p\omega(t) - pz^*(t) = p\omega(t)$ for a given TUCE price p. Otherwise $z^*(t) = 0$, and since $F_l(0) = 0$, the player gets $p\omega(t)$. Clearly, this implies that for any coalition $S \subset T$, the payoff it gets is $\sum_l \int_{A_l \cap S} p\omega(t) d\lambda(t) = p\mu(S)$ by definition of $\mu(S)$.

Next, since F_l are differentiable on $\mathbb{R}^M_{++}$, $\nabla f(\mu(T))$ is well defined. It is left to show that the TUCE price p is $= \nabla f(\mu(T)) \in \mathbb{R}^M_{++}$. Indeed, first, in a symmetric TUCE allocation (almost) all agents of the same type chose the same allocation (which is true in any TUCE if F_l are strictly quasiconcave), denote them by z_l^* respectively. Since all types get a strictly positive payoff in equilibrium and the allocation is Pareto for every l $z_l^* \lambda(A_l)$ solves

$$\max_{\{y_l\}_l, y_l \in \mathbb{R}^M_+} G(y, W) \equiv \sum_l F_l(y_l) \colon \sum_l y_l \leq W$$

Since the value of this problem is by definition $f(\mu(T))$, for every l such that $z_l^* \in \mathbb{R}^M_{++}$, we have $\nabla f(\mu(T)) = \nabla F_l(z_l^* \lambda(A_l))$, which is strictly positive since F_l is strictly increasing on $\mathbb{R}^M_{++}$. Further, since F is CRS, $\nabla F_l(\lambda(A_l) z_l^*) = \nabla F_l(z_l^*)$, which is equal to p if $z_l^* \in \mathbb{R}^M_{++}$ by consumer optimisation. □

3.8. Asymptotic Shapley value for a simple duopoly game.
Consider the following simple duopoly. There are two types of players. The small players are of the first type and their total endowment is ω_0. Let $T_0 = [0, 1]$ and let λ_0 be the uniform measure. Each of the two big players (a_1, a_2) has endowment ω_1, they are of equal size: $\lambda_1(a_i) = 1$. Let $F \colon \mathbb{R}^M_+ \to \mathbb{R}$, absolutely continuous, be the "best" production function (with the highest output) available to the players.

Proposition 1. *The asymptotic Shapley value for a simple duopoly game (with absolutely continuous production) allocates $\int_0^1 \big[F(t\omega_0 + \omega_1) + t(F(t\omega_0 + 2\omega_1) - 2F(t\omega_0 + \omega_1))\big] dt$ to any big player.*

Proof. Any (finite) game in an admissible sequence should have n small players and two big ones. Each of the small players holds $\frac{1}{n}\omega_0$. The (marginal) contribution of any big player to a coalition of k small players is $h_1(\frac{k}{n})$, where $h_1\colon t \mapsto F(t\omega_0 + \omega_1)$; while the contribution to a coalition already containing a big player and k small ones is $h_2(\frac{k}{n})$, $h_2\colon t \mapsto F(t\omega_0 + 2\omega_1) - F(t\omega_0 + \omega_1)$.

There are $(n + 2)!$ possible permutations of all players. For any position k of the first big player there are $(n + 2 - k)n!$ permutations where the second big player follows the first, while in the rest $((k-1)n!)$ the opposite is true. Therefore Shapley value for a big player a_i is

$$\begin{aligned}[\phi V_n](\{a_i\}) &= \tfrac{n!}{(n+2)!}\Big[\textstyle\sum_{k=2}^{n+2}(k-1)h_2\big(\tfrac{k-2}{n}\big) + \sum_{k=1}^{n+1}(n+2-k)h_1\big(\tfrac{k-1}{n}\big)\Big] \\ &= \textstyle\sum_{k=2}^{n+2} \tfrac{1}{n+1}\tfrac{k-1}{n+2}h_2\big(\tfrac{k-2}{n}\big) + \sum_{k=1}^{n+1} \tfrac{1}{n+1}(1 - \tfrac{k}{n+2})h_1\big(\tfrac{k-1}{n}\big)\end{aligned}$$

To establish the claim it is sufficient to show that the first summand converges to $\int_0^1 th_2(t)dt$ and the second one converges to $\int_0^1(1-t)h_1(t)dt$ as $n \to \infty$.

We demonstrate the latter in detail, the proof of the former is analogous and is left to the reader. Let $g_n(t) = (1 - \frac{nt+1}{n+2})h_1(t)$, then the second term can be written as a Lebesgue integral $\int_0^1 z_n(t)d\mu$ of a simple function z_n returning a constant value $g_n(\frac{k-1}{n})$ on $]\frac{k-1}{n}, \frac{k}{n}]$ for $k \in \{1, n+1\}$, by definition of Lebesgue integral, since

$$\int_0^1 z_n(t)d\mu = \sum_{k=1}^{n+1} \frac{1}{n+1} g_n(\frac{k-1}{n}) \leq h_1(1) < \infty \quad \forall n$$

It is then left to show that z_n converges uniformly on $[0, 1]$ to the function $t \mapsto (1 - t)h_1(t)$. Indeed, by triangular inequality,

$$\sup_t |(1-t)h_1(t) - z_n(t)| \leq \sup_t |g_n(t) - (1-t)h_1(t)| + \sup_t |z_n(t) - g_n(t)|$$

The first term can be made arbitrarily small for large enough n:

$$\sup_t \left|(\frac{nt+1}{n+2} - t)h_1(t)\right| \leq \frac{2}{n+2}h_1(1)$$

For the second term it follows from absolute continuity of F (and hence h_1) and thus g_n on $[0, 1]$. Thus, the uniform convergence. □

Corollary 4. *Let $F(x, y) = \sqrt{xy}$ and $\omega_0 = (x, 0), \omega_1 = (0, y)$ then the Shapley value for the big player is above his* TUCE *payoff.*

Proof. The TUCE payoff of a big player is equal to $\nabla F(\omega_0 + 2\omega_1)\omega_1$ in the economy with the simple duopoly. Using the specification of F, this payoff is $\frac{1}{2\sqrt{2}}\sqrt{xy}$, which is smaller than the Shapley value calculated using proposition 1, $\sqrt{xy}(\frac{2}{3} - (2 - \sqrt{2})\frac{2}{5}))$. □

4. Comments and a Conjecture

To restate: our main result is that if there is more than one big player in a smooth economy, the asymptotic nucleolus assigns to every big player an allocation with the same value as his competitive payoff. One could interpret the result as reflecting the fierce competition among the big players (e.g., à la Bertrand). However, Shapley value might assign a better allocation to a big player than his competitive payoff, and we conjecture this is true in general for an economy with several big players (of the same type) and a smooth production.

Note that the asymptotic Shapley value in prop. 1 for a simple duopoly corresponds to a heuristic based on the result of [4] showing that the asymptotic Shapley value for scalar games of bounded variation (converging to a mixed game) is one of the values of the mixed game characterized by [7],[2] such that the 'location' of every big player (in a 'permutation') is uniformly and independently distributed on the unit interval. Indeed, then for the mixed game, the Shapley value allocates $\int_0^1 \int_t^1 h_1(t)dzdt + \int_0^1 \int_0^t h_2(t)dzdt = \int_0^1 [(1-t)h_1(t) + th_2(t)]dt$ to a big player. The asymptotic Shapley value to the syndicate (a single big) player in [6] is also consistent with this heuristic calculation, and it can be obtained directly using the similar argument to the proof of prop. 1, which should be extendible to any number of big players of the same type rather easily. Thus the main challenge in proving the conjecture is showing that the inequality asserted in cor. 4 holds in general.

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