

Fairness in Bargaining and the Kalai-Smorodinsky Solution

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Abstract

A bargaining solution guarantees minimal equity if each player's payoff is at least as large as the minimum of the payoffs assigned to him by the equal-gain (i.e., egalitarian) and equal-loss solutions. The Kalai-Smorodinsky solution is the unique scale-invariant 2-person solution with this property. There does not exist a scale-invariant n -person solution with this property.

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1 Introduction

Fairness is basic consideration in bargaining. It involves lines of reasoning in the spirit of (i) if you gain this much I should gain at least this much, and (ii) if I sacrifice (or, alternatively, invest) this much, you should sacrifice (invest) this much. A fairness-oriented arbitrator may therefore wish to accommodate both of these principles. Unfortunately, however, he may find himself facing a problem, since (i) and (ii) typically lead to different recommendations. More formally, the first is expressed by the *egalitarian solution* (due to Kalai (1977)), the second is expressed by the *equal loss solution* (Chun (1988)), and the two bargaining solutions typically differ.

Moreover, even if our arbitrator could decide between these competing principles, he would still face a problem, since either of the aforementioned solutions violate *scale invariance*; that is, under either of these solutions, the choice of the units that measure player i 's payoff affects the payoff received by players other than i .

Can the arbitrator balance between (i) and (ii) ? Can he do so in a way consistent with scale invariance? I argue that in the 2-person bargaining problem the arbitrator should employ the Kalai-Smorodinsky bargaining solution (due to Kalai and Smorodinsky (1975)). In the multi-person case, on the other hand, I argue that the arbitrator does not have a compelling solution to resort to.

The reason is this. Pretend for the moment that the problem faced by our arbitrator is actually easier than it is, and the violation of scale invariance is not an issue: the only question is how to pick, or merge, two perfectly kosher principles: (i) vs. (ii). More formally, the arbitrator needs to pick either the egalitarian solution or the equal loss solution, or combine them somehow. It

seems a moderate starting point to demand that each player’s payoff be at least as large as the minimum of the payoffs assigned to him by the egalitarian solution and the equal-loss solution. That is, denoting these solutions by E and EL respectively, and denoting a generic solution by μ and a generic bargaining problem by S , the requirement is $\mu_i(S) \geq \min\{E_i(S), EL_i(S)\}$ for every player i and every problem S . This requirement is met, trivially, by both E and EL , but there may be additional solutions that satisfy it. We can now bring scale invariance back into the picture and ask whether there are scale invariant such solutions. It turns out that in the 2-person case there is one and only one such solution—that of Kalai and Smorodinsky’s. In the multi-person case, on the other hand, there does not exist such a solution.

The rest of this note is organized as follows. Section 2 lays down the model, Sections 3 and 4 contain the results—for 2-person and multi-person bargaining respectively—and Section 5 concludes.

2 Model

The following model is due to Nash (1950). An n -player bargaining problem is defined to be a compact, convex, and comprehensive set $S \subset \mathbb{R}_+^n$ such that $\mathbf{0} \equiv (0, \dots, 0) \in S$.¹ It consists of all the (v-N.M) utility allocations that the players can achieve via cooperation: if they agree on $x \in S$, then player i receives the utility payoff x_i ; if they fail to reach a unanimous agreement, all get zero. Let \mathcal{B} denote the collection of all bargaining problems. A *bargaining solution* (a solution, for short) is any function $\mu: \mathcal{B} \rightarrow \mathbb{R}_+^n$ that satisfies $\mu(S) \in S$ for all $S \in \mathcal{B}$. A solution μ is *scale invariant* if for every linear transformation

¹Comprehensiveness means that if $x \in S$ and y is such that $\mathbf{0} \leq y \leq x$, then $y \in S$ (vector inequalities: $v \leq w$ if and only if $v_i \leq w_i$ for all i).

λ and every $S \in \mathcal{B}$, it is the case that $\mu(\lambda \circ S) = \lambda \circ \mu(S)$.² One solution that satisfies this property is the Kalai-Smorodinsky solution, KS , due to Kalai and Smorodinsky (1975). For every $S \in \mathcal{B}$, $KS(S)$ is the highest point $x \in S$ according to the standard partial order on \mathbb{R}_+^n that satisfies $x = \theta a(S)$ for some $\theta \in \mathbb{R}_+$, where $a_i(S)$ —the *ideal payoff* of player i in problem S —is defined by $a_i(S) \equiv \max\{y_i | y \in S\}$. The following solutions, on the other hand, E and EL , are not scale invariant. For every $S \in \mathcal{B}$, $E(S)$ is the highest point $x \in S$ according to the standard partial order on \mathbb{R}_+^n that satisfies $x_i = x_j$ for all i and j , and $EL(S)$ is the highest point $x \in S$ according to the standard partial order on \mathbb{R}_+^n that satisfies $a_i(S) - x_i = a_j(S) - x_j$ for all i and j .

Definition 1. A bargaining solution, μ , **guarantees minimal equity**, if for each $S \in \mathcal{B}$ and each player i , $\mu_i(S) \geq \min\{E_i(S), EL_i(S)\}$.

3 2-person bargaining

Theorem 1. Let $n = 2$. A scale invariant solution, μ , guarantees minimal equity if and only if $\mu = KS$.

Proof. I start by proving that KS guarantees minimal equity. Let $S \in \mathcal{B}$. Let $a \equiv a(S)$, let $k \equiv KS(S)$, and let $x \equiv EL(S)$. If $a_1 = a_2$ then $k = E(S) = EL(S)$ and we are done. Suppose then, wlog, that $a_1 > a_2$. Obviously, we can also assume, wlog, that $a_1 = 1$. In this case $k_1 > E_1(S)$; I will prove that $k_2 \geq x_2$.

Assume by contradiction that $k_2 < x_2$. Since $\frac{k_2}{k_1} = \frac{a_2}{a_1} = a_2$ we have $k_2 = a_2 k_1$. Therefore $a_2 k_1 < x_2$. By the definition of EL , $a_1 - x_1 = a_2 - x_2$,

²The function $\lambda: \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ is a *linear transformation* if $\lambda \circ (x_1, \dots, x_n) \equiv (\lambda_1 x_1, \dots, \lambda_n x_n)$, where $\lambda_i > 0$ for all i .

hence $x_2 = a_2 + x_1 - a_1 = a_2 + x_1 - 1$. Therefore $a_2 + x_1 - 1 > a_2 k_1 \Rightarrow a_2(1 - k_1) > 1 - x_1$, and since $a_2 < a_1 = 1$, $1 - k_1 > 1 - x_1$, or $x_1 > k_1$. We obtain that $x_i > k_i$ for both $i \in \{1, 2\}$, in contradiction to the strong Pareto optimality of KS .³

I now turn to prove uniqueness. Let μ be a scale-invariant solution that guarantees minimal equity. Let $S \in \mathcal{B}$. Let λ be the linear transformation given by $\lambda \circ x = (\frac{a_2(S)}{a_1(S)}x_1, x_2)$. Note that $E(T) = EL(T) = KS(T)$, where $T \equiv \lambda \circ S$. Since μ guarantee minimal equity, $\mu(T) \geq KS(T)$ and therefore $\mu(T) = KS(T)$. Since both μ and KS are scale invariant, $\mu(S) = KS(S) = \lambda^{-1} \circ E(T)$. \square

4 Multi-person bargaining

As the following theorem shows, the 2-person case is special: in general, scale invariance is inconsistent with minimal equity.

Theorem 2. *Let $n \geq 3$. There does not exist a scale invariant solution that guarantees minimal equity.*

Sketch of proof: Let $n \geq 3$. Assume by contradiction that there exists a solution with the aforementioned properties, μ . Let $S \in \mathcal{B}$. Consider the linear transformation λ which is given by $\lambda_i = \frac{a_1(S)}{a_i(S)}$, and let $T \equiv \lambda \circ S$. By the arguments from the proof of Theorem 1, $\mu(T) = KS(T)$ and hence $\mu(S) = KS(S)$. However, KS does not guarantee minimal equity when $n \geq 3$ (it is not hard to come up with examples that show this fact; I omit such an example for the sake of brevity). \square

³A solution μ is *strongly Pareto optimal* if $\mu(S) \in P(S) \equiv \{x \in S : (y \geq x) \& (y \neq x) \Rightarrow y \notin S\}$ for every $S \in \mathcal{B}$.

5 Conclusion

Fairness presents two difficulties to bargaining theory: there are two natural fairness principles, equating gains and equating losses, each of which is built on the idea of interpersonal utility comparisons. The first challenge is how to account for both of these principles simultaneously, and the second is how to do that without invoking interpersonal comparisons.

In this note I have showed that, in a formal sense, these challenges cannot be overcome in general. In the special case of 2-person bargaining, however, they can: the Kalai-Smorodinsky solution, which is interpersonal-comparisons-free, provides a middle-ground “between” these principles in the 2-person case.

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6 References

- [1] Chun, Y., (1988), “The equal-loss principle for bargaining problems,” *Economic Letters*, **26**, 103-106.
- [2] Kalai, E., and Smorodinsky, M. (1975), “Other solutions to Nash’s bargaining problem,” *Econometrica*, **43**, 513-518.
- [3] Kalai, E. (1977), “Proportional Solutions to Bargaining Situations: Interpersonal Utility Comparisons,” *Econometrica*, **45**, 1623-1630.
- [4] Nash, J.F, (1950), “The Bargaining Problem,” *Econometrica*, **18**, 155-162.