

# Fairness, Efficiency, and the Nash Bargaining Solution

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## Abstract

A bargaining solution balances fairness and efficiency if each player's payoff lies between the minimum and maximum of the payoffs assigned to him by the egalitarian and utilitarian solutions. In the 2-person bargaining problem, the Nash solution is the unique scale-invariant solution satisfying this property. Additionally, a similar result, relating the weighted egalitarian and utilitarian solutions to a weighted Nash solution, is obtained. These results are related to a theorem of Shapley, which I generalize. For  $n \geq 3$ , there does not exist any  $n$ -person scale-invariant bargaining solution that balances fairness and efficiency.

*Keywords:* Bargaining; fairness; efficiency; Nash solution;

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# 1 Introduction

Consider a set of players (bargainers) who are facing a bargaining problem. Fairness and efficiency are natural objectives that they (or an arbitrator) may have in mind, but taking both of them into account simultaneously is non-trivial, because promoting the one typically involves compromising on the other. Moreover, this task involves an additional difficulty: both fairness and efficiency rely on the idea of *interpersonal utility comparisons*. The former involves considerations in the spirit of “if you gain this much I should gain at least this much,” the latter involves considerations in the spirit of “do me a favor, it would only cost you a little, but would help me a lot.” How do we know that the utility functions in terms of which the bargaining problem is defined capture those interpersonal comparisons appropriately? One may argue that we need to have the “right” utilities before any further analysis of fairness and efficiency is to be carried out.

Let us say that a 2-person bargaining problem is *harmonic* if its egalitarian and utilitarian solutions agree. Given a bargaining problem, we can rescale its utilities such that the resulting problem is harmonic. Defining these new utilities to be the “right” ones resolves both of the issues described in the previous paragraph. First, we obtain a utility scale to work with; second, the tension between fairness and efficiency is trivially resolved, because the bargaining problem which is defined by this utility scale is harmonic. Shapley (1969) showed that the egalitarian/utilitarian solution of the rescaled problem, when scaled back, is the Nash solution of the original problem. Therefore, Shapley’s Theorem can be thought of as describing a sense in which fairness and

efficiency are reconciled, and showing that only the Nash solution satisfies it.<sup>1</sup> This sense implicitly assumes that the bargaining solution is invariant to independent (linear) rescalings of the players' utilities.

I provide a simpler sense in which the Nash solution reconciles fairness and efficiency. Simpler—since it does not refer to the utility scales. I demand that the solution lies “between” the egalitarian and utilitarian solutions; namely, that for every bargaining problem and every player the following is true: the player's solution payoff lies between the minimum and the maximum of the payoffs assigned to him by the egalitarian and utilitarian solutions.<sup>2</sup> I call this *balancing fairness and efficiency*. I prove that the Nash solution balances fairness and efficiency. It follows from Shapley's Theorem that it is the unique scale-invariant solution with this property.

The utilitarian and egalitarian solutions have straightforward generalizations to nonsymmetric bargaining: that of the former is obtained by maximizing a weighted sum of utilities, and that of the latter—by assigning payoffs according to fixed (not necessarily identical) proportions. These solutions can be thought of in terms of a two-step procedure: first, utilities are rescaled—player 1's payoff is scaled by  $p \in (0, 1)$  and player 2's payoff is scaled by  $1 - p$ ; next, either egalitarianism or utilitarianism is applied. Given the weights  $(p, 1 - p)$ , a bargaining solution *balances fairness and efficiency with respect to*

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<sup>1</sup>Harsanyi (1959) has a lemma that states that the Nash solution is the only utility allocation that coincides simultaneously with both the egalitarian and utilitarian solutions for some rescaling of the utilities; it follows from the scale-invariance of the Nash solution that the egalitarian/utilitarian solution of the scaled problem, when scaled back, is the Nash solution of the original problem. Shapley (1969) was the first to state the result in this way.

<sup>2</sup>More precisely, the requirement is that there is a selection from the utilitarian solution (which, in general, is multi valued), such that the above condition is satisfied. See Section 3 below for the precise definition.

$p$  if each player’s solution payoff lies between the minimum and the maximum of the payoffs assigned to him by the corresponding weighted egalitarian and utilitarian solutions. I show that there exists a function  $h: (0, 1) \rightarrow (0, 1)$  that satisfies  $h(\frac{1}{2}) = \frac{1}{2}$ , such that the following is true: given  $p \in (0, 1)$ , the weighted Nash solution with weights  $(h(p), 1-h(p))$  is the unique scale-invariant solution that balances fairness and efficiency with respect to  $p$ . Based on this result, I obtain the following generalization of Shapley’s Theorem: any problem can be rescaled such that the  $p$ -weighted egalitarian and utilitarian solutions of the resulting problem agree, and scaling the agreed-upon point back results in the  $h(p)$ -weighted Nash solution of the original problem. It is worth noting that the function  $h$  satisfies  $(p - \frac{1}{2})(h(p) - p) > 0$  for all  $p \neq \frac{1}{2}$ , which means that in order to balance fairness and efficiency with respect to  $p$ , the strong player—the one whose payoff gets more weight—needs to be favored, in the sense of being assigned to an augmented weight in the Nash product.

The relationship between egalitarianism and utilitarianism has for quite some time been the subject of a vibrant discourse, especially since the publication of Rawls’ *A Theory of Justice*, back in 1971 (see Arrow (1973), Harsanyi (1975), Lyons (1972), Sen (1974), and Yaari (1981), among others). A particularly heated debate sprouted up between Rawls and Harsanyi (see Rawls (1974), which was replayed to by Harsanyi (1975)), the former advocating the maxmin rule as the “right” principle for governing society’s decisions, the latter advocating the sum-of-utilities criterion. Within the confines of 2-person bargaining theory, my paper proposes a compromise between these competing positions;<sup>3</sup> moreover, subject to scale-invariance, this compromise—the Nash

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<sup>3</sup>Of course, the theory of distributive justice concerns itself also with issues outside the bargaining model, such as *freedom*, *needs*, and more (see Roemer (1986)). I do not claim that the current paper proposes a general reconciliation between egalitarianism and

solution—is unique. To put it in a catchy phrase: only Nash can bridge the gap between Harsanyi and Rawls.

Whether this compromise is a decent compromise will be discussed at the end of the paper. As already seen above, the answer to this question is far from obvious, since without symmetry balancing fairness and efficiency implies a bias in favor of the strong and against the weak, which, to say the least, is not the first thing that comes to mind when thinking about distributive justice.

The aforementioned results do not extend to multi-person bargaining: given any  $n \geq 3$ , there does not exist a scale-invariant  $n$ -person bargaining solution that lies “between” the egalitarian and utilitarian solutions.

The rest of the paper is organized as follows. Section 2 describes the formal model. Section 3 presents the main concept of interest—balancing fairness and efficiency. It also introduces a related concept—*guarantee of minimal fairness*—and discusses the relation between the two. Section 4 considers symmetric 2-person bargaining, Section 5 introduces asymmetry, Section 6 considers multiperson bargaining, Section 7 concludes, and the Appendix collects proofs which are omitted from the text.

## 2 Model

An  $n$ -person bargaining problem is a pair  $(S, d)$  such that  $S \subset \mathbb{R}^n$  is closed and convex, and  $d \in S$  is such that  $S_d \equiv \{x \in S \mid x > d\}$  is nonempty and bounded.<sup>4</sup> The points of  $S$ , the *feasible set*, are the (v.N-M) utility vectors that the players can achieve via cooperation (if they agree on  $(x_1, \dots, x_n) \in S$ , utilitarianism—it does so only in the context of a particular model. It is, however, an important model.

<sup>4</sup>Vector inequalities in  $\mathbb{R}^n$ :  $xRy$  if and only if  $x_iRy_i$  for all  $i$ , for both  $R \in \{>, \geq\}$ .

player  $i$  receives the utility payoff  $x_i$ ), and  $d$  specifies their utilities in case they do not agree unanimously on some point of  $S$ ; it is called the *disagreement point*. Let  $\mathcal{B}_n$  denote the collection of all such pairs.

A *solution* on  $\mathcal{B}_n$  is any function  $\mu: \mathcal{B}_n \rightarrow \mathbb{R}^n$  that satisfies  $\mu(S, d) \in S$  for all  $(S, d) \in \mathcal{B}_n$ . The *Nash solution* (due to Nash (1950)),  $N$ , is the unique maximizer of  $\prod_{i=1}^n (x_i - d_i)$  over  $x \in S_d$ . The *egalitarian solution* (due to Kalai (1977)),  $E$ , is given by  $E(S, d) = d + \epsilon \cdot \mathbf{1}$ ,<sup>5</sup> where  $\epsilon$  is the maximal number such that the right hand side is in  $S$ . Given a problem  $(S, d)$ , let  $\mathbb{U}(S, d) \equiv \operatorname{argmax}_{x \in S_d} \sum x_i$ . A bargaining solution,  $\mu$ , is *utilitarian*, if  $\mu(S, d) \in \mathbb{U}(S, d)$  for every problem  $(S, d)$ . A generic utilitarian solution is denoted by  $U$ . I will sometimes abuse terminology a little, and refer to  $U$  as *the* utilitarian solution.

A solution,  $\mu$ , is *weakly Pareto optimal* if  $\mu(S, d) \in WP(S) \equiv \{x \in S | y > x \Rightarrow y \notin S\}$  for every  $(S, d) \in \mathcal{B}_n$ ; it is *strongly Pareto optimal* if the analogous condition holds when  $WP(S)$  is replaced by  $P(S) \equiv \{x \in S | y \neq x \& y \geq x \Rightarrow y \notin S\}$ ; it is *scale-invariant* if  $\lambda \circ \mu(S, d) = \mu(\lambda \circ S, \lambda \circ d)$  for every positive linear transformation  $\lambda: \mathbb{R}^n \rightarrow \mathbb{R}^n$  and every  $(S, d) \in \mathcal{B}_n$ ;<sup>6</sup> I will sometime call a positive linear transformation a *rescaling*.

Let  $\mathcal{B}_n^+ \subset \mathcal{B}_n$  consist of those  $(S, d) \in \mathcal{B}_n$  such that (i)  $S \subset \mathbb{R}_+^n$  and (ii)  $d = \mathbf{0}$ . In the sequel, the domain of analysis will be  $\mathcal{B}_n^+$ . With the disagreement point normalized to the origin, I will abuse notation a little and denote a problem solely by its feasible set,  $S$ . Accordingly, I use the notation  $\mathbb{U}(S) \equiv \operatorname{argmax}_{x \in S} \sum x_i$ . Let  $\mathcal{B}_n^U = \{S \in \mathcal{B}_n^+ | \mathbb{U}(S) \text{ is a singleton}\}$ . Let  $\mathcal{B}_2^*$  be the collection of those problems in  $\mathcal{B}_2^+$  which are *smooth*: those  $S \in \mathcal{B}_2^+$  for

<sup>5</sup> $\mathbf{1} = (1, \dots, 1)$ . Similarly,  $\mathbf{0} = (0, \dots, 0)$ .

<sup>6</sup>A function  $\lambda: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a positive linear transformation if  $\lambda \circ (x_1, \dots, x_n) \equiv (\lambda_1 x_1, \dots, \lambda_n x_n)$  for some numbers  $\lambda_i > 0$

which  $WP(S) = P(S) = \{(a, f(a)) | a \in [0, A]\}$ , where  $A > 0$  is some number and  $f: [0, A] \rightarrow \mathbb{R}_+$  is a twice differentiable strictly concave function. The family  $\mathcal{B}_2^*$  is dense in  $\mathcal{B}_2^+$ : for every  $S \in \mathcal{B}_2^+$  there exists a sequence  $\{S_n\} \subset \mathcal{B}_2^*$ , such that  $S_n$  converges to  $S$  in the Hausdorff metric. Moreover,  $\mathcal{B}_2^* \subset \mathcal{B}_2^U$ , and if  $\{S_n\} \subset \mathcal{B}_2^*$  is a sequence that converges to  $S$  in the Hausdorff metric, then  $\lim_n U(S_n) \in \mathbb{U}(S)$ , independent of whether  $S \in \mathcal{B}_2^U$ .

A solution,  $\mu$ , is *continuous*, if for every sequence of problems in its domain  $\{S_n\}$  and every problem in its domain  $S$ ,  $\mu(S_n)$  converges to  $\mu(S)$  if  $S_n$  converges to  $S$  in the Hausdorff metric. The solutions  $N$  and  $E$  are continuous on  $\mathcal{B}_n^+$ , and  $U$  is continuous on the restricted domain  $\mathcal{B}_n^U$ .

### 3 Balancing fairness and efficiency

The main concept of interest in this paper is this:

**Definition 1.** A solution on  $\mathcal{B}_n^+$ ,  $\mu$ , **balances fairness and efficiency** if for every  $S \in \mathcal{B}_n^+$  there exists a  $U(S) \in \mathbb{U}(S)$  such that the following is true for every  $i$ :

$$\min\{E_i(S), U_i(S)\} \leq \mu_i(S) \leq \max\{E_i(S), U_i(S)\}.$$

Definition 1 intends to express a form of compromise between fairness and efficiency. Implicitly, it identifies “betweenness,” in the simple sense of ordering numbers on the real line, as the appropriate notion for such a compromise. It is logically stronger than the following:

**Definition 2.** A solution on  $\mathcal{B}_n^+$ ,  $\mu$ , **guarantees minimal fairness** if for every  $S \in \mathcal{B}_n^+$  there exists a  $U(S) \in \mathbb{U}(S)$  such that the following is true for every  $i$ :

$$\mu_i(S) \geq \min\{E_i(S), U_i(S)\}.$$

Underlying this definition is not a notion of compromise (or betweenness), but a one of insurance: a solution that adheres to it guarantees that payoffs will never fall short of a certain bound, this bound incorporating both fairness and efficiency. In the special case of two players, the two definitions are equivalent; with more than two players, the former is strictly stronger.

**Proposition 1.** *Let  $n = 2$ . Then, a solution balances fairness and efficiency if and only if it guarantees minimal fairness.*

**Proposition 2.** *Let  $n \geq 3$ . Then, there exists a solution that guarantees minimal fairness, but that does not balance fairness and efficiency.*

## 4 Symmetric 2-person bargaining

As the following proposition shows, the 2-person Nash solution adheres to Definition 1.

**Proposition 3.** *The Nash solution balances fairness and efficiency on  $\mathcal{B}_2^+$ .*

*Proof.* We need to prove that  $\mu = N$  satisfies the requirement of Definition 1. By the continuity properties of the bargaining solutions, it is enough to establish this fact on the restricted domain  $\mathcal{B}_2^*$ .

Assume by contradiction that there exists a problem  $S \in \mathcal{B}_2^*$  for which this is not true. Let  $f$  be the smooth function describing  $P(S) = WP(S)$ . Since  $U(S) \in P(S)$ , we can assume, wlog, that  $U_1(S) \geq E_1(S)$ . If  $N(S)$  is not between  $E(S)$  and  $U(S)$ , then either  $N_1(S) > U_1(S)$  or  $N_1(S) < E_1(S)$ .

Suppose first that  $N_1(S) > U_1(S)$ . Note that  $N(S)$  is the solution to the maximization of  $af(a)$  and  $U(S)$  is the solution to the maximization of  $a + f(a)$ , both over  $a \in [0, A]$ . The first order condition for  $N$  is  $f(N_1(S)) +$

$N_1(S)f'(N_1(S)) = 0 \Rightarrow f'(N_1(S)) = -\frac{f(N_1(S))}{N_1(S)}$ . The derivative of the objective that  $U$  maximizes is  $1 + f'(a)$ , and at the optimum (i.e., at  $a = U_1(S)$ ) it is nonpositive, because, by assumption,  $U_1(S) < N_1(S) < A$ . Therefore,  $f'(U_1(S)) \leq -1$ . Since  $f$  is concave,  $N_1(S) > U_1(S)$  implies  $f'(N_1(S)) \leq f'(U_1(S))$ . Therefore  $-\frac{f(N_1(S))}{N_1(S)} \leq -1$ , or  $f(N_1(S)) \geq N_1(S)$ . Therefore,  $N(S) = (N_1(S), f(N_1(S))) \geq (N_1(S), N_1(S)) > (U_1(S), U_1(S))$ , in contradiction to  $U(S) \in P(S)$ .

Suppose, on the other hand, that  $N_1(S) < E_1(S) \equiv e$ . This implies that  $N(S) = (e - x, e + y)$  for some  $x, y > 0$ , because  $N(S) \in P(S)$ . Next, I argue that  $x \geq y$ . To see this, assume by contradiction that  $x < y$ , so  $N_1(S) + N_2(S) > E_1(S) + E_2(S)$ . Also, recall that  $U_1(S) \geq E_1(S) = e$ . If  $U_1(S) > e$  then there exists an  $\alpha \in (0, 1)$  such that  $\alpha U(S) + (1 - \alpha)N(S) > (e, e) = E(S)$ . Therefore  $U_1(S) = e$ , so  $U(S) = (e, e)$ , because  $E(S) \in WP(S) = P(S)$ . By definition of  $U$ ,  $2e \geq 2e - x + y$ . Therefore  $x \geq y$ , in contradiction to the initial assumption  $x < y$ . Thus, it must be that  $x \geq y$ . Finally, note that by definition of  $N$ ,  $(e - x)(e + y) > e^2$ , hence  $ey > ex + xy$ . Combining this inequality with  $x \geq y$  gives  $ex \geq ex + xy$ , a contradiction.

□

Next, one would like to know whether there are other solutions on  $\mathcal{B}_2^+$  that balance fairness and efficiency. The trivial answer to this question is that there are infinitely many such solutions, as any selection between  $E$  and  $U$  will do. This question becomes more interesting if one introduces the additional restriction of scale-invariance. Under this restriction, it turns out, only  $N$  balances fairness and efficiency. To prove this uniqueness, the following result, which is due to Shapley (henceforth, Shapley's Theorem), is useful.

**Theorem 1.** (Shapley (1969)) *Let  $S \in \mathcal{B}_2^+$  and  $x \in S$ . Then  $x = N(S)$  if and*

only if the following statement is true:

- There exists a rescaling of  $S$ ,  $T = \lambda \circ S$ , such that  $\lambda \circ x = E(T) \in \mathbb{U}(T)$ .

In particular, it follows that one can always rescale a given problem as to obtain a harmonic one—a one whose egalitarian and utilitarian solutions agree (because  $N(S)$  exists for every  $S$ ).

**Corollary 1.**  *$N$  is the unique scale-invariant solution on  $\mathcal{B}_2^+$  that balances fairness and efficiency.*

*Proof.* Let  $\mu$  be a solution that balances fairness and efficiency and let  $S \in \mathcal{B}_2^+$ . By Shapley's Theorem, there exists a rescaling of  $S$  such that the rescaled problem, call it  $T$ , satisfies  $E(T) = (x, x) \in \mathbb{U}(T)$ . Suppose first that  $\mathbb{U}(T)$  is a singleton; then its unique element is  $U(T) = (x, x)$ . In this case  $\mu_i(T) \geq \min\{E_i(T), U_i(T)\} = x$ , and it follows from the strong Pareto optimality of  $U$  that  $\mu(T) = (x, x)$ . By Proposition 3,  $N(T) = (x, x)$ . Since both  $\mu$  and  $N$  are scale-invariant solutions,  $\mu(S) = N(S)$ .

Consider now the case where  $\mathbb{U}(T)$  is not a singleton. Let  $U(T)$  be the selection from  $\mathbb{U}(T)$  such that the requirement of Definition 1 holds. If  $U(T) = (x, x)$ , then the proof is completed by the same argument as above. Suppose, on the other hand that  $U(T) \neq (x, x)$ . By Shapley's Theorem  $(x, x) = E(T) \in P(T)$ , and therefore  $U_i(T) < x$  and  $U_j(T) > x$  for some  $(i, j)$ . Wlog, suppose that  $(i, j) = (1, 2)$ . That is, the utilitarian selection  $U$  is to the left of  $(x, x)$  and the segment connecting the two has a slope  $-1$ . Moreover, the solution point  $\mu(T)$  belongs to this segment. I argue that it must be that  $\mu(T) = (x, x)$ . To see this, assume by contradiction that it is to the left of  $(x, x)$ . Now, rescale player 1's utility by some  $\lambda > 1$  close to 1. By scale-invariance the solution point changes only slightly, but the utilitarian solution of the scaled problem

jumps to the right of the 45° line—a contradiction. Finally, the latter argument also applies to the Nash solution, and therefore  $N(T) = \mu(T) = (x, x)$ .  $\square$

## 5 2-person bargaining without symmetry

Given the weights  $(p, 1 - p) > \mathbf{0}$ , the corresponding weighted solutions to the problem  $S \in \mathcal{B}_2^+$  are defined as follows: a weighted utilitarian solution maximizes the sum  $px_1 + (1 - p)x_2$  over  $x \in S$ , the weighted egalitarian solution is given by  $(p\epsilon, (1 - p)\epsilon)$ , where  $\epsilon$  is the maximal number such that the latter expression belongs to  $S$ , and the weighted Nash solution maximizes the product  $x_1^p x_2^{(1-p)}$  over  $x \in S$ . I will denote the weighted egalitarian and Nash solutions by  $E^p$  and  $N^p$ , respectively. Let  $\mathbb{U}^p(S) \equiv \operatorname{argmax}_{x \in S} px_1 + (1 - p)x_2$  and let  $\theta \equiv \frac{p}{1-p}$ . Note that given  $S \in \mathcal{B}_2^+$ ,  $E^p(S)$  takes the form  $(\theta y, y)$ ,  $N^p$  maximizes the product  $x_1^\theta x_2$  over  $x \in S$ , and every solution that picks points in  $\mathbb{U}^p(S)$  maximizes  $\theta x_1 + x_2$  over  $x \in S$ .

The following is an adaptation of Definition 1 to the symmetry-free 2-person setting.

**Definition 3.** *Let  $p \in (0, 1)$ . A solution on  $\mathcal{B}_2^+$ ,  $\mu$ , **balances fairness and efficiency with respect to  $p$**  if for every  $S \in \mathcal{B}_2^+$  there exists an  $U^p(S) \in \mathbb{U}^p(S)$ , such that the following is true for every  $i$ :*

$$\min\{E_i^p(S), U_i^p(S)\} \leq \mu_i(S) \leq \max\{E_i^p(S), U_i^p(S)\}.$$

Let:

$$h(p) = \frac{p^2}{2p^2 - 2p + 1}.$$

**Theorem 2.** *Let  $p \in (0, 1)$  and let  $\mu$  be a scale-invariant solution. Then  $\mu$  balances fairness and efficiency with respect to  $p$  if and only if  $\mu = N^{h(p)}$ .*

Since  $h(\frac{1}{2}) = \frac{1}{2}$ , Theorem 2 implies both Proposition 3 and Corollary 1; its proof (which appears in the Appendix) is based on the following lemma:

**Lemma 1.** *Let  $p \in (0, 1)$  and  $S \in \mathcal{B}_2^+$ . Then there is a rescaling of  $S$ ,  $T = \lambda \circ S$ , such that  $E^p(T) \in \mathbb{U}^p(T)$ .*

Moreover, when this lemma is combined with the other ideas that are utilized in Theorem 2's proof, the following result obtains:

**Theorem 3.** *(A generalized Shapley Theorem) Let  $S \in \mathcal{B}_2^+$ ,  $x \in S$ , and  $p \in (0, 1)$ . Then  $x = N^{h(p)}(S)$  if and only if the following statement is true:*

- *There exists a rescaling of  $S$ ,  $T = \lambda \circ S$ , such that  $\lambda \circ x = E^p(T) \in \mathbb{U}^p(T)$ .*

*Proof.* Let  $p \in (0, 1)$ . Fix  $S$  and  $x \in S$ . Suppose first that  $x = N^{h(p)}(S)$ . Let  $\lambda$  be the rescaling from Lemma 1. I will prove that  $\lambda \circ x = E^p(T)$ , where  $T = \lambda \circ S$ . That is, I will prove that  $N^{h(p)}(T) = E^p(T)$ . Assume by contradiction that  $N^{h(p)}(T) \neq E^p(T)$ ; wlog, since both of these points are strongly Pareto optimal in  $T$ , suppose that  $N^{h(p)}$  is to the left and above  $E^p(T)$ . Let  $\beta = \frac{h(p)}{1-h(p)}$ . Note that  $\beta = \theta^2$ , where  $\theta = \frac{p}{1-p}$ . Let  $N^{h(p)} = (x, y)$ . Note that  $E^p(T) = (\theta z, z)$  for some  $z$ . This means that  $\frac{y}{x} > \frac{1}{\theta}$ , and since the tangency condition associated with  $N^{h(p)}(T)$  is  $\beta \frac{y}{x} = \theta$ , it follows that the negative of the slope of the hyperbola associated with  $N^{h(p)}$  at the point  $(x, y)$  is greater than  $\theta$ . This, however, is incompatible with the fact that  $E^p(T)$  is down and to the right of  $(x, y)$ , and the slope there is only  $\theta$ .

Conversely, suppose that there exists a rescaling  $\lambda$  such that  $T = \lambda \circ S$  satisfies  $\lambda \circ x = E^p(T) \in \mathbb{U}^p(T)$ . We need to prove that  $x = N^{h(p)}(S)$ . Assume by contradiction that  $x \neq N^{h(p)}(S)$ . Applying the linear transformation  $\lambda$  to both sides gives  $\lambda \circ x \neq N^{h(p)}(\lambda \circ S)$ . That is,  $E^p(T) \neq N^{h(p)}(T)$ . Wlog, suppose that  $N^{h(p)}(T)$  is to the left of  $E^p(T)$ . The arguments from the previous paragraph complete the proof.  $\square$

## 6 Multiperson bargaining

The following result shows that, under the restriction to scale-invariant solutions, one cannot balance fairness and efficiency, in the sense of Definition 1, when there are more than two players. It is straightforward to modify the proof in order to show that one cannot balance fairness and efficiency, in a sense analogous to that of Definition 3, with respect to any  $(p_1, \dots, p_n) > 0$ .

**Proposition 4.** *There does not exist a scale-invariant solution on  $\mathcal{B}_n^+$  that balances fairness and efficiency, for any  $n \geq 3$ .*

*Proof.* Assume by contradiction that there exists a scale-invariant solution on  $\mathcal{B}_n^+$ , for some  $n \geq 3$ , that balances fairness and efficiency. Given  $a, b > 0$ , let:

$$S_{ab} \equiv \{(ax_1, bx_2, x_3, \dots, x_n) | (x_1, \dots, x_n) \in \mathbb{R}_+^n, \sum x_i^2 \leq 1\}.$$

It is straightforward that  $E_i(S_{ab}) = \sqrt{\frac{1}{\frac{1}{a^2} + \frac{1}{b^2} + n - 2}}$  for all  $i$ . In particular, this is true for  $i = 3$ . Additionally, it is clear that  $U_i(S_{ab}) \rightarrow 0$  as  $a \rightarrow \infty$ , for all  $i \neq 1$ . Since  $E_3(S_{ab}) \approx \sqrt{\frac{1}{\frac{1}{b^2} + n - 2}}$  for all sufficiently large  $a$ 's, it follows that  $E_3(S_{ab}) > U_3(S_{ab})$  for all sufficiently large  $a$ 's. Therefore, since  $\mu$  balances fairness and efficiency, the following must hold for all sufficiently large  $a$ 's:

$$\mu_3(S_{ab}) \leq E_3(S_{ab}). \quad (1)$$

Since  $\mu$  is scale-invariant,  $\mu_3(S_{ab}) = \mu_3(S_{11})$ , and since it balances fairness and efficiency,  $\mu_3(S_{11}) = \sqrt{\frac{1}{n}}$ . Plugging the expressions for  $\mu_3(S_{ab})$  and  $E_3(S_{ab})$  into (1) gives that the following must hold for all sufficiently large  $a$ 's:

$$\sqrt{\frac{1}{n}} \leq \sqrt{\frac{1}{\frac{1}{a^2} + \frac{1}{b^2} + n - 2}}.$$

Taking  $a \rightarrow \infty$  gives:

$$\sqrt{\frac{1}{n}} \leq \sqrt{\frac{1}{\frac{1}{b^2} + n - 2}}. \quad (2)$$

Note, however, that (2) is violated for all  $b \in (0, \sqrt{\frac{1}{2}})$ .

□

Proposition 4 reinforces a well-known pattern: there is a difference between 2-person and multi-person bargaining, in the sense that there are results which are true (false) in the former setting, but false (true) in the latter.<sup>7</sup>

## 7 Conclusion

In this paper I have introduced the notion of *balancing fairness and efficiency* in bargaining. This concept is generally stronger than the related *guarantee of minimal fairness*, though in the 2-person case they coincide. Restricting attention to scale-invariant solutions, I have shown that this balancing is impossible in multi-person bargaining, and that there is a unique way to achieve it in 2-person bargaining: by applying Nash’s solution. The balancing concept assumes symmetric players, but it is generalized straightforwardly to a concept that relates “weighted egalitarianism” and “weighted utilitarianism.”

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<sup>7</sup>Here is an example of a possibility result for the 2-person case that cannot be generalized to more players: Perles and Maschler (1981) derived the existence (and uniqueness) of a 2-person bargaining solution which, in addition to satisfying other standard axioms, is *super additive* (see their paper for the definition of this axiom); subsequently, Perles (1982) proved that no such solution exists in the 3-person case. Here is an example for a possibility result for more than two players that does not hold for two players: in the paper cited earlier in the Introduction, Shapley proved that there does not exist an ordinal, efficient, and symmetric 2-person bargaining solution, but he constructed a 3-person solution with these properties; Samet and Safra (2005) generalized the construction to  $n$  players.

Similarly to the symmetric case, it has been shown that a certain weighted Nash solution is the only scale-invariant solution that adheres to this concept.

In the existing literature, the result which is closest in spirit to the analysis carried out here is that of Moulin (1983), who characterized the Nash solution by two axioms only: Nash’s IIA and *midpoint domination*—an axiom that requires each player’s solution payoff to lie above the average of his disagreement and ideal payoffs (the latter being his maximal payoff in the individually rational part of the feasible set).<sup>8</sup> In other words, it says that the solution should Pareto dominate “randomized dictatorship”: by letting each player be a dictator with equal probability—an event in which he obtains his ideal payoff—one cannot improve on the solution. The fair lottery in the randomized dictatorship process is a starting point that guarantees a minimal degree of fairness; from there on, efficiency enters the picture.

So, is the compromise that the Nash solution proposes between fairness and efficiency an acceptable one? One may very well argue that the answer is negative. As was already noted by Luce and Raiffa (1958, p.129-130), the Nash solution tends to favor players with utility functions closer to linearity, and this, in the words of Menahem Yaari, “can be regarded as a bias in favor of the rich and against the poor.”<sup>9</sup> A similar bias has presented itself in the current paper: when the weight on player 1’s utility is  $p \in (0, 1)$ , the only way to balance egalitarianism and utilitarianism (in a way consistent with scale-invariance) is by applying the nonsymmetric Nash solution that puts weight

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<sup>8</sup>Anbarci (1998) improved Moulin’s result by weakening midpoint domination; his characterization replaces it by an axiom that expresses the same requirement, but applies only to triangular feasible sets. de Clippel (2007) also derives a two-axiom characterization of the Nash solution, one of the axioms being midpoint domination (the other is *disagreement convexity*; see his paper for the definition).

<sup>9</sup>Yaari (1981), p.38.

$h(p)$  on player 1's utility. Since  $(p - \frac{1}{2})(h(p) - p) > 0$  for all  $p \neq \frac{1}{2}$ , the strong player's weight is augmented in the Nash product and the weak player's weight is discounted.

There is an additional, and even simpler sense, in which the Nash solution is “more utilitarian than egalitarian”. Whenever the Nash solution coincides with the egalitarian solution it also coincides with the utilitarian solution. This is a simple geometric feature of the Nash solution: note that the slope of the parabola  $x_2 = \frac{c}{x_1}$  is  $-\frac{c}{x_1^2}$ , hence equals  $-1$  at  $x_2 = x_1$ , and therefore, whenever the Nash and the egalitarian solutions agree, the agreed-upon point maximizes the sum of the players' utilities. The “converse” is, of course, not true, as can be seen, for example, in rectangular feasible sets.

To summarize, in bridging the gap between egalitarianism and utilitarianism, the Nash solution constitutes a biased compromise. This may lead one to reject it. A person holding such a view may argue that in order to promote fairness—with or without regard for efficiency—scale-invariance should be excluded: a joint utility scale, with respect to which *all* bargaining problems are to be solved, must be specified in advance. On the other hand, a person who insists on the v.N-M axioms of utility theory will consequently insist on scale-invariance. In this case, the aforementioned compromise can be viewed either as an unfortunate “second-best,” or, alternatively, as an *ethical* conclusion. A person holding this view may argue that favoring the rich is (in some circumstances, at least) the ethical thing to do. Both views are a matter of interpretation, and it is left for the reader to decide where she stands.

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## 8 Appendix

*Proof of Proposition 1:* Let  $\mu$  be a 2-person solution that guarantees minimal fairness. I will prove that the requirement of Definition 1 is satisfied for *any* selection  $U$  out of  $\mathbf{U}$ . Let  $U$  be such a selection. Assume by contradiction that there is an  $S$  and an  $i$  such that  $\mu_i(S) > \max\{U_i(S), E_i(S)\}$ . Wlog, suppose that  $i = 1$ . Since  $U(S) \in P(S)$ ,  $\mu_2(S) < U_2(S)$ . Therefore,  $\mu_2(S) = E_2(S) \equiv y$ . Since  $\mu_1(S) > y$ ,  $E(S) = (y, y) \notin P(S)$ . I argue that  $(a, b) \in S$  implies  $b \leq y$ . To see this, assume by contradiction that there is an  $(a, b) \in S$  with  $b > y$ . Note that  $\mu(S) = (x, y)$  for some  $x > y$ . Therefore, we can find an  $\alpha \in (0, 1)$  sufficiently close to one, such that  $\alpha\mu(S) + (1 - \alpha)(a, b) > (y, y)$ , in contradiction to  $E(S) \in WP(S)$ . Now, since  $U(S) \neq (x, y)$  and since  $U(S)$  maximizes the sum of utilities in  $S$ ,  $U_1(S) < x$  implies that  $U_2(S) > y$ —a contradiction.  $\square$

*Proof of Proposition 2:* Let  $n \geq 3$ . Let  $S_n^* = \text{conv hull}\{\mathbf{0}, ie_i | i = 1, \dots, n\}$ , where  $\{e_i | i = 1, \dots, n\}$  is the standard basis for  $\mathbb{R}^n$ . Note that  $U(S_n^*) = ne_n$  and  $E(S_n^*) = \sum \alpha_i ie_i$ , where  $\{\alpha_i\}$  are convex weights that satisfy  $l\alpha_l = m\alpha_m$  for all  $1 \leq l, m \leq n$ . Define the solution  $\mu_n^*$  as follows:

$$\mu_n^*(S) = \begin{cases} \frac{1}{n} \sum ie_i & \text{if } S = S_n^* \\ E(S) & \text{otherwise} \end{cases}$$

I will prove that (i) this solution guarantees minimal fairness, and (ii) that it does not balance fairness and efficiency. For both (i) and (ii), clearly, only the problem  $S_n^*$  needs to be considered. Requirement (i) is obviously satisfied for players  $i < n$ , because the utilitarian payoff for each of them is zero. Thus, what needs a proof here is that player  $n$ 's payoff is at least as large as his

egalitarian payoff (because the latter is obviously smaller than his utilitarian payoff, which equals  $n$ ). Assume by contradiction that this is not the case; i.e., that  $1 < \alpha_n n$ . Since  $l\alpha_l = m\alpha_m$  for all  $1 \leq l, m \leq n$ , it follows that  $\alpha_1 > 1$ , in contradiction to the fact that  $(\alpha_1, \dots, \alpha_n)$  are convex weights.

Next, consider (ii). I will prove that the  $n - 1$ -th player receives more than the maximum of his egalitarian and utilitarian payoffs. That is, that  $\frac{n-1}{n} > (n-1)\alpha_{n-1}$ . To see this, assume by contradiction that  $\alpha_{n-1} \geq \frac{1}{n}$ . Now, since  $i\alpha_i = (n-1)\alpha_{n-1} \geq \frac{n-1}{n}$  for every  $i$ , we have that

$$\sum_{i=1}^n \alpha_i \geq \frac{n-1}{n} \sum_{i=1}^n \frac{1}{i} \equiv F(n).$$

To obtain the contradiction, I will show that  $F(n) > 1$  for all  $n \geq 3$ . This is certainly the case, since  $F(3) = \frac{2}{3}(1 + \frac{1}{2} + \frac{1}{3}) = \frac{2}{3} \cdot \frac{11}{6} = \frac{22}{18} > 1$ , and  $F$  is strictly increasing (it is a product of two strictly increasing functions of  $n$ ).  $\square$

*Proof of Theorem 2:* I start with the “if” part. By the continuity arguments invoked in Proposition 3, we can restrict attention to  $\mathcal{B}_2^*$ . Let  $S \in \mathcal{B}_2^*$ . Let  $f$  be the smooth function describing  $S$ 's boundary. Let  $\theta = \frac{p}{1-p}$  and  $\beta = \frac{h(p)}{1-h(p)}$ . Note that  $\beta = \theta^2$ . Assume by contradiction that  $N^{h(p)}(S)$  is not in between  $U^p(S)$  and  $E^p(S)$ .

Case 1:  $U_1^p(S) \leq E_1^p(S)$ .

There are two possibilities: either  $N_1^{h(p)}(S) < U_1^p(S)$  or  $N_1^{h(p)}(S) > E_1^p(S)$ . Start by assuming the former. Letting  $a$  denote the payoff for player 1, we see that the tangency condition associated with  $N^{h(p)}$  is  $\beta \frac{f(a)}{a} = -f'(a)$ . Since  $N^{h(p)}$  is, by assumption, to the left of  $U^p$ ,  $-f'(a) < \theta$ ; combining this with  $\beta = \theta^2$  we obtain  $a > \theta f(a)$ , which contradicts  $N_1^{h(p)}(S) < E_1^p(S)$ . Next, consider  $N_1^{h(p)}(S) > E_1^p(S)$ . Again, denoting by  $a$  player 1's payoff under  $N^{h(p)}$  we have  $a > \theta f(a)$ , and therefore  $-f'(a) = \beta \frac{f(a)}{a} < \beta \frac{1}{\theta} = \theta$ , in contradiction

to  $N_1^{h(p)}(S) > U_1^p(S)$ .

Case 2:  $U_1^p(S) > E_1^p(S)$ .

There are two possibilities: either  $N_1^{h(p)}(S) > U_1^p(S)$  or  $N_1^{h(p)}(S) < E_1^p(S)$ . Start by assuming the former. Since  $N^{h(p)}$  is to the right of  $U^p$ ,  $-f'(a) > \theta$ , hence  $\beta \frac{f(a)}{a} = \theta^2 \frac{f(a)}{a} > \theta$ . Therefore,  $\theta f(a) > a$ , in contradiction to  $N_1^{h(p)}(S) > E_1^p(S)$ . Next, consider  $N_1^{h(p)}(S) < E_1^p(S)$ . This means that  $a < \theta f(a)$  and therefore  $-f'(a) = \beta \frac{f(a)}{a} = \theta^2 \frac{f(a)}{a} > \theta$ . This means that  $N^{h(p)}$  must lie to the right of  $U^p$ , in contradiction to  $N_1^{h(p)}(S) < E_1^p(S) < U_1^p(S)$ .

I now turn to uniqueness. Let  $\mu$  be an arbitrary solution with the aforementioned properties, and let  $p \in (0, 1)$  be the parameter with respect to which  $\mu$  balances fairness and efficiency. Let  $S \in \mathcal{B}_2^+$ . By Lemma 1,  $S$  can be rescaled such that the resulting problem, call it  $T$ , satisfies  $E^p(T) \equiv (\theta y, y) = U^p(T)$  for some  $U^p(T) \in \mathbb{U}^p(T)$ , where  $\theta = \frac{p}{1-p}$ . Suppose first that  $\mathbb{U}^p(T)$  is a singleton, containing only  $(\theta y, y)$ . In this case  $U^p(T) = E^p(T) = \mu(T) = N^{h(p)}(T)$ ,<sup>10</sup> and by scale-invariance,  $\mu(S) = N^{h(p)}(S)$ .

Suppose that  $\mathbb{U}^p(T)$  is not a singleton, and let  $U^p(T)$  be the selection out of it that satisfies the requirement of Definition 3. If  $U^p(T) = (\theta y, y)$ , then the proof is completed by the same argument as above. Suppose, on the other hand that  $U^p(T) \neq (\theta y, y)$ . By Lemma 1,  $(\theta y, y) = E^p(T) \in P(T)$ , and therefore  $U_i^p(T) < E_i^p(T)$  and  $U_j(T) > E_j^p(T)$  for some  $(i, j)$ . Wlog, suppose that  $(i, j) = (1, 2)$ . That is, the selection  $U^p$  is to the left of  $E^p(T)$  and the segment connecting the two has a slope  $-\theta$ . Moreover, the solution point  $\mu(T)$  belongs to this segment. I argue that it must be that  $\mu(T) = E^p(T)$ . To see this, assume by contradiction that it is to its left. Now, rescale player 1's utility by some  $\lambda > 1$  close to 1. By scale-invariance the solution point changes only

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<sup>10</sup>The last equality here is due to the fact that we just proved that  $N^{h(p)}$  balances fairness and efficiency with respect to  $p$ .

slightly, but the weighted utilitarian solution of the scaled problem jumps to the right of the  $E^p$ —a contradiction. Finally, the latter argument also applies to the weighted Nash solution, and therefore  $N^{h(p)}(T) = \mu(T)$ . By scale-invariance  $N^{h(p)}(S) = \mu(S)$ .  $\square$

*Proof of Lemma 1:* Let  $p \in (0, 1)$  and  $\theta = \frac{p}{1-p}$ . It is easy to see that it suffices to prove the lemma for problems in  $\mathcal{B}_2^*$ . Let then  $S$  be such a problem and let  $f$  be the smooth function, defined on  $[0, A]$ , which describes its boundary. Since both  $U^p$  and  $E^p$  are *homogeneous*—namely,  $\mu(cS) = c\mu(S)$  for every  $S$ ,  $c > 0$ , and  $\mu \in \{U^p, E^p\}$ —it suffices to consider rescalings of one player’s utility. Wlog, I will consider rescalings of player 2’s utility by  $\lambda > 0$ . With  $E^p(T) = (a, \lambda f(a))$  for some  $a \in [0, A]$ , the required equalities are  $\theta \lambda f(a) = a$  and  $\lambda f'(a) = -\theta$ . That is, it is sufficient (and necessary) to find an  $a \in [0, A]$  such that  $\frac{a}{f(a)\theta} = \frac{-\theta}{f'(a)}$ , or  $\psi(a) \equiv \frac{-af'(a)}{f(a)} = \theta^2$ . There exists a unique such  $a$  because, by the assumptions on  $f$ , the function  $\psi$  is strictly increasing, and satisfies  $\psi(0) = 0$  and  $\psi(a) \rightarrow \infty$  as  $a \rightarrow A$ .  $\square$

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