SEPARATE CONTROL OVER THE LOCAL AND THE ASYMPTOTIC BEHAVIOUR IN L_p SPACES

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Abstract. We introduce 2 parameter variants $L_{p,q}$ of the Lebesgue spaces, to gain separate control on the asymptotic behaviour (p) and the local behaviour (q). Thus they behave w.r.t. p like the spaces ℓ_p and w.r.t. q like the spaces L_q on a probability space. Convolution behaves very well on those spaces.

1. The spaces $L_{p,q}$

Definition 1. *G is a locally compact Abelian group, with Haar measure λ, M is the space of bounded measures on G, $\mathcal{K}$ is the increasing filtering family of compact subsets of G, and C_0 is the Banach space of continuous functions tending to 0 at ∞.*

(i) Given a relatively compact measurable subset B of G with non-empty interior, $L_{p,q} \stackrel{\text{def}}{=} \{f \text{ measurable}\colon G \to \overline{\mathbb{R}} \mid \|f\|_{p,q} \stackrel{\text{def}}{=} \|x \mapsto \|\mathbb{1}_{x+B} f\|_q\|_p < \infty\}$, mod null functions, for $1 \le p, q \le \infty$.
(ii) For $1 \le p \le \infty$, C_p is the subspace of continuous functions in $L_{p,\infty}$.
(iii) $t_h\colon t \mapsto t + h$ is the translation by h on G; and $\mathbb{S}_h\colon f \mapsto f \circ t_{-h}$ the shift on functions on G.

$\complement X$ is the complement of a subset X and $\#X$ its cardinality. ϵ_x is the unit mass at x.

Example. Take $G = \mathbb{R}$, with B the unit interval. Functions in $L_{\infty,1}$ are then 'uniformly' locally integrable. If $f \ge 0$ is unimodal, $\|f\|_{1,\infty} = \|f\|_1 + \|f\|_\infty$, so $L_{1,\infty}$ contains most classical probability densities.

Remark 1. $L_{\infty,1}$ (L_1^{loc} for μ with compact support) is the natural function space on which M acts by convolution (rem. 5, prop. 2). These spaces allow in [2] to study as operators the derivatives of a fixed point with infinite-dimensional parameters.

Theorem 1. *In the following statements all the constants implied by norm equivalence are independent of p, q.*

(i) $L_{p,q}$ is a Banach lattice, and $\mathbb{S}_h$ an isometry on $L_{p,q}$.

Norm equivalences:

(ii) Two different relatively compact measurable sets B_0 and B_1 with non-empty interior yield equivalent norms $\|\cdot\|_{p,q}$.
(iii) Let $B \subseteq G$ be measurable and relatively compact, and $J \subseteq G$ be uniformly discrete, i.e., s.t. $(J - J) \cap V = \{0\}$ for some neighbourhood V of

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0. *If $J+B$ covers G up to a negligible set, then* $\|j \mapsto \|\mathbb{1}_{j+B}f\|_q\|_p$ *is an equivalent norm, where the p-norm denotes the ℓ_p norm over J.*

(iv) There exist pairs (J,B) as sub iii, with B a difference of 2 compact Baire sets, with non-void interior, and s.t. its translates by J form a partition of G.

(v) $\|f\|_{\infty,q} = \mathsf{sup}_x \|\mathbb{1}_{x+B}f\|_q$.

Monotonicity w. r. t. p and q:

(vi) $\|\cdot\|_{p,p} = \lambda(B)^{1/p}\|\cdot\|_p$, *so* $\|\cdot\|_{p,p}$ *is equivalent to* $\|\cdot\|_p$.

(vii) $q < q' \Rightarrow \|\cdot\|_{p,q} \leq (\lambda(B))^{1/q-1/q'}\|\cdot\|_{p,q'} \leq \mathsf{max}\{1,\lambda(B)\}\|\cdot\|_{p,q'}$.

(viii) $p' > p \Rightarrow \|\cdot\|_{p',q} \leq \|\cdot\|_{p,q}$ *up to a constant factor.*[1]

(ix) C_p *is closed in* $L_{p,\infty}$, *and, if* $p < \infty$, *injects continuously into* C_0.

Monotonicity w. r. t. f:

(x) The $\overline{\mathbb{R}}$-valued measurable functions form a complete lattice. Denote by $\mathsf{ess\,sup}$ *the* sup *in this lattice. For a filtering increasing net* $f_\alpha \geq 0$, $\|\mathsf{ess\,sup}\, f_\alpha\|_{p,q} = \mathsf{sup}\|f_\alpha\|_{p,q}$.

(xi) $\forall f \in L_{p,q}, \mathsf{lim}_{\varepsilon\searrow 0}\|\varepsilon \wedge |f|\|_{p,q} = 0$.

(xii) If $p < \infty$, $\forall f \in L_{p,\infty} \exists g \in C_0 : |f| \leq g$ *except on a null set.*

Hölder and duality:

(xiii) If $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ *and* $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$, $\|fg\|_{p,q} \leq \|f\|_{p_1,q_1}\|g\|_{p_2,q_2}$ *for f, g Lebesgue-measurable.*
Conversely, $N(f) = \mathsf{sup}_{\|g\|_{p_2,q_2}\leq 1}\|fg\|_{p,q}$ *is equivalent to* $\|f\|_{p_1,q_1}$.

(xiv) Up to norm-equivalence, for $\frac{1}{p}+\frac{1}{p'} = \frac{1}{q}+\frac{1}{q'} = 1$, $L_{p',q'}$ *is a closed subspace of the dual of* $L_{p,q}$, *and equals this dual if* $p < \infty$ *and* $q < \infty$.

Proof. i: Note first that for $f \in L_{p,q}$, $|f| < \infty$ a.e.: else there would be a compact set of positive measure where $|f| = \infty$, implying that $\|\mathbb{1}_{x+B}f\|_q = \infty$ on some open set. Thus $L_{p,q}$ is a vector lattice. $\|\cdot\|_{p,q}$ is a norm because $\|\cdot\|_p$ and $\|\cdot\|_q$ are so. Completeness follows then by showing, using the monotone convergence theorem, that a norm-summable series converges. Shift-invariance is clear.

ii: Let $g_i(x) = \|\mathbb{1}_{x+B_i}f(s)\|_q$, $i \in \{0,1\}$. The translates of the interior of B_0 cover G, so for some finite subset $I \subseteq G$, $I + B_0$ covers the (compact) closure of B_1, so $\mathbb{1}_{x+B_1} \leq \sum_{y\in I}\mathbb{1}_{y+x+B_0}$. Then, a first use of the triangle inequality yields $g_1(x) \leq \sum_{y\in I} g_0(x+y)$. A second one yields then immediately $\|g_1\|_p \leq \sum_{y\in I}\|x \mapsto g_0(x+y)\|_p, = \#I\|g_0\|_p$ by shift invariance.

iii: New norm is weaker: By uniform discreteness of J, and continuity of addition, take a compact neighbourhood B' of 0 s.t. $(B'-B')\cap(J-J) = \{0\}$. Then all translates of B' by $j \in J$ are disjoint: if $j_i \in J$ s.t. $(j_1+B')\cap(j_2+B') \neq \emptyset$, then $j_1 - j_2 \in (B'-B')\cap J = \{0\}$. Thus $\|f\|_{p,q} \overset{\text{def}}{=} \|x \mapsto \|\mathbb{1}_{x+B-B'}f\|_q\|_p \geq \|x \mapsto \sum_J \mathbb{1}_{j+B'}(x)\|\mathbb{1}_{x+B-B'}f\|_q\|_p\,, \geq \|x \mapsto \sum_J \mathbb{1}_{j+B'}(x)\|\mathbb{1}_{j+B}f\|_q\|_p$ since $j + B \subseteq x + B - B'$ $\forall j \in J, \forall x \in j + B'$. And this equals $\lambda(B')^{1/p}\|j \mapsto \|\mathbb{1}_{j+B}f\|_q\|_p$, whether $p = \infty$ or not. Thus the claim, since $\lambda(B')^{1/p} \geq \mathsf{min}\{1,\lambda(B')\} > 0$.

New norm is stronger: first, using instead of B another measurable relatively compact subset B' yields a weaker norm. Indeed, the translates of B by J cover G up to a null set, so the set of such translates by $I = (\bar{B}'-\bar{B})\cap J$ covers B' a.e.; but I, being compact and discrete, is finite, so the rest of the proof of ii applies.

Then replace first B by one of its compact neighbourhoods. The union of its translates is closed, B being compact and J closed; its open complement is negligible, thus empty: $J+B = G$. For the norm with J use $B' = B+B$. Then, as above, $\|f\|_{p,q} \leq \|x \mapsto \sum_J \mathbb{1}_{j+B}(x)\|\mathbb{1}_{x+B}f\|_q\|_p \leq \|x \mapsto \sum_J \mathbb{1}_{j+B}(x)\|\mathbb{1}_{j+B'}f\|_q\|_p = \lambda(B')^{1/p}\|j \mapsto \|\mathbb{1}_{j+B'}f\|_q\|_p$. Conclude by $\lambda(B')^{1/p} \leq \mathsf{max}\{1,\lambda(B)\}$.

[1] Unlike in ℓ_p spaces, the constant factor can not be taken as 1, even for $G = \mathbb{R}$ and $B = [0,1]$. If $f = \mathbb{1}_{[0,1]}$, then $\|f\|_{p,q} = \big(2/(1+p/q)\big)^{\frac{1}{p}}$, which is > 1 for $p < q$, $= 1$ for $p = q$ or ∞, and < 1 else. E.g., $\|f\|_{3,1} = \frac{1}{\sqrt[3]{2}} < .8$, so even 1.25 would not suffice.

iv: By [3, p. 110], G has an open subgroup $G_1 = G_0 \times \mathbb{R}^n$, where G_0 is a compact subgroup. Let $B = G_0 \times [0,1[^n$, $J_0 = \{0\} \times \mathbb{Z}^n$; select some z_i in each coset i of G_1, and let $J = \bigcup_i (z_i + J_0)$: clearly the $j + B$ ($j \in J$) partition G, and the trace of $J - J$ on G_1 equals J_0, so J is uniformly discrete.

x: Complete lattice: define the $\operatorname{ess\,sup}$ locally, using a *concassage* (a disjoint family $K_\alpha \in \mathcal{K}$ with negligible complement s.t. K_α is the support of the restriction of λ to K_α). With $f = \operatorname{ess\,sup} f_\alpha$, clearly $\|f\|_{p,q} \geq \sup \|f_\alpha\|_{p,q}$. Conversely, note first that $x \mapsto \|\mathbb{1}_{x+B} f\|_q = \operatorname{ess\,sup} \|\mathbb{1}_{x+B} f_\alpha\|_q$: indeed, given $K \in \mathcal{K}$, let $K_1 = K + \bar{B}$ (with $\bar{B}$ the closure of B); on the compact set K_1, $f = \sup f_{\alpha_n}$, where the α_n are an increasing sequence; thus also $\|\mathbb{1}_{x+B} f\|_q = \sup \|\mathbb{1}_{x+B} f_{\alpha_n}\|_q = \operatorname{ess\,sup} \|\mathbb{1}_{x+B} f_{\alpha_n}\|_q \leq \operatorname{ess\,sup} \|\mathbb{1}_{x+B} f_\alpha\|_q$ on K. Since this holds for every K, $\|\mathbb{1}_{x+B} f\|_q \leq \operatorname{ess\,sup} \|\mathbb{1}_{x+B} f_\alpha\|_q$, and since $x \mapsto \|\mathbb{1}_{x+B} f\|_q$ is measurable, clearly$\|\mathbb{1}_{x+B} f\|_q \geq \operatorname{ess\,sup} \|\mathbb{1}_{x+B} f_\alpha\|_q$, thus equality. The result follows then by taking $\|\cdot\|_p$ norms.

v: For $q < \infty$, $\|\mathbb{1}_{x+B} f\|_q$ is lower semi-continuous in x [use prop. 2 (which does not depend on this point) with $\mu = \epsilon_x$ and a sequence $g_n \geq 0$ in L_∞ increasing to $|f|^q$; alternatively, it is the usual continuity property for L_p spaces], and thus its $\operatorname{ess\,sup}_x$ is equal to its $\sup_x$.

If $q = \infty$, $\|f\|_\infty \geq \sup_x \|\mathbb{1}_{x+B} f\|_\infty$. By definition, $\|f\|_\infty = \sup_C \|\mathbb{1}_C f\|_\infty$, C compact. For any C there is its element x_C such that $\|\mathbb{1}_C f\|_\infty = \|\mathbb{1}_{N_{x_c}} f\|_\infty$ for any open neighbourhood N_{x_c} of x_c in C. B has non-empty interior B^0, so $U_c = x_c - B^0$ is open, and for any $x \in U_c$, $\|\mathbb{1}_{x+B} f\|_\infty \geq \|\mathbb{1}_C f\|_\infty$. Thus $\operatorname{ess\,sup}_x \|\mathbb{1}_{x+B} f\|_\infty \geq \|\mathbb{1}_C f\|_\infty$ for any C, and hence $\operatorname{ess\,sup}_x \|\mathbb{1}_{x+B} f\|_\infty \geq \sup_C \|\mathbb{1}_C f\|_\infty$ Combining the inequalities above, $\operatorname{ess\,sup}_x \|\mathbb{1}_{x+B} f\|_\infty \geq \|f\|_\infty \geq \sup_x \|\mathbb{1}_{x+B} f\|_\infty$, but $\operatorname{ess\,sup}(\cdot) \leq \sup(\cdot)$ then establishes the equality: $\operatorname{ess\,sup}_x \|\mathbb{1}_{x+B} f\|_\infty = \|f\|_\infty = \sup_x \|\mathbb{1}_{x+B} f\|_\infty$.

vi: The case of $q = \infty$ is covered above, and the proof for $q < \infty$ follows from shift invariance: by Fubini, $\iint \mathbb{1}_{x+B} f(s) ds dx = \lambda(B) \int f(x) dx$, whenever $f \geq 0$ and B are measurable. Finally, note $\min\{1, \lambda(B)\} \leq \lambda(B)^{1/p} \leq \max\{1, \lambda(B)\}$.

xi: If $p = \infty$, the result is immediate. Else, $\|\mathbb{1}_{x+B}(\varepsilon \wedge |f|)\|_q \to 0$, hence the result by the monotone convergence theorem.

xiii: If the right hand side is ∞, there is nothing to prove; else if one of its factors is 0, fg is negligible and the result holds. Else both factors are finite, and one concludes using Hölder's inequality twice.

For the converse, use, with the equivalent norm described in iv, the corresponding result for L_p spaces twice, first on each $z + B$ ($z \in J$) to choose g there up to scale, next on J to fix those scaling factors.

vii: Use xiii with $g = 1$, $p_2 = \infty, q_1 = q'$.

viii: Let $\|f\|^*_{p,q} = \|x \mapsto \|\mathbb{1}_{x+B-B} f\|_q\|_p$; using ii $\|f\|^*_{p,q} \leq k\|f\|_{p,q}$ for some $k > 0$. Let $F_x = \|\mathbb{1}_{x+B} f\|_q$: for $y \in x + B$, $\|\mathbb{1}_{y+B-B} f\|_q \geq F_x$, so $F_x \leq \lambda(B)^{-1/p} \|f\|^*_{p,q}$, hence $\|f\|_{\infty,q} \leq K\|f\|_{p,q}$, where $K = k \max\{1, \lambda(B)^{-1}\}$. Thus, if $\|f\|_{p,q} = K^{-1}$, $F_x \leq 1$, so, for $p' > p$, $F_x^{p'} \leq F_x^p$: $\int F_x^{p'} dx \leq \|f\|^p_{p,q} = K^{-p}$, thus $\|f\|_{p',q} \leq K^{-p/p'}$, hence by homogeneity $\|\cdot\|_{p',q} \leq K^{1-p/p'} \|\cdot\|_{p,q} \leq \max\{1, K\} \|\cdot\|_{p,q}$.

xii: By viii, $\|f\|_{\infty,\infty} \leq K\|f\|_{p,\infty}$: f is bounded. Hence this reduces to: if further $f = \mathbb{1}_S$, then S is relatively compact up to a null set (indeed, let then $v_n = 2^{-n}\|f\|_\infty$, $S_n = \{\, x \mid |f(x)| \geq v_n \,\}$: if $g_n \in C_0$, $v_{n-1}\mathbb{1}_{S_n} \leq g_n \leq v_{n-1}$ a.e., then $g = \sum_{n \geq 0} g_n \in C_0$ and $g \geq f$ a.e.). With a norm as sub iv, $Z = \{z \in J \mid \|\mathbb{1}_{z+B} f\|_\infty = 1\}$ must be finite, since the ℓ_p norm of $\mathbb{1}_Z$ is finite; each $z + B$ being relatively compact, the result follows.

ix: follows from xii and viii.

xiv: The first part follows from xiii: the inequality implies that $L_{p',q'}$ maps into $L^*_{p,q}$, and the converse that this map is a norm-equivalence (thus, injective).

For surjectivity, let $\varphi \in L^*_{p,q}$, and use norms as sub iv. Apply first φ to the restriction of $L_{p,q}$ to each $z + B$ ($z \in J$): this yields a measurable function ψ. Each compact set C being covered by finitely many such translates, ψ is such that $\forall C$ compact, $\psi \mathbb{1}_C \in L_{q'}$ and $\forall f \in L_{p,q}$, $\varphi(\mathbb{1}_C f) = \int \psi \mathbb{1}_C f$.

If $q > 1$, let $\chi_z = \mathbb{1}_{z+B}\, \mathsf{sign}(\psi) \big(\frac{|\psi|}{\|\psi \mathbb{1}_{z+B}\|_{q'}}\big)^{q'/q}$, where the ratio is set to $(\lambda(B))^{-1/q'}$ if the denominator is 0. If $q = 1$, let $C_z = \{x \in z + B \mid |\psi(x)| \geq (1-\varepsilon)\|\psi \mathbb{1}_{z+B}\|_\infty\}$ and $\chi_z = \mathsf{sign}(\psi)\frac{\mathbb{1}_{C_z}}{\lambda(C_z)}$. Note that, $\forall z$, $\|\chi_z\|_q = 1$ and $\int \psi \chi_z \geq (1-\varepsilon)\|\psi \mathbb{1}_{z+B}\|_{q'}$.[2]

Let $f = \sum_J \alpha_z \chi_z$ for some $\alpha \in \ell_p(J)$: $\|f\|_{p,q} = \|\alpha\|_p$. Assume $\alpha \geq 0$ with finite support. Then $|\varphi(f)| \leq \|\varphi\|\|f\|_{p,q} = \|\varphi\|\|\alpha\|_p$, and, by the above, $\varphi(f) = \sum_J \alpha_z \int \psi \chi_z \geq (1-\varepsilon)\sum_J \alpha_z \|\psi \mathbb{1}_{z+B}\|_{q'}$: $\sum_J \alpha_z \|\psi \mathbb{1}_{z+B}\|_{q'} \leq \|\varphi\|\|\alpha\|_p$, $\varepsilon > 0$ being arbitrary. This extends immediately, first to arbitrary $\alpha \geq 0$, then to any $\alpha \in \ell_p$. Thus $\|z \mapsto \|\psi \mathbb{1}_{z+B}\|_{q'}\|_{p'} \leq \|\varphi\|$, by the duality theorem for ℓ_p spaces if $p < \infty$, and trivially if $p = \infty$: $\|\psi\|_{p',q'} \leq \|\varphi\|$.

In particular, by xiii, $\zeta\colon f \mapsto \varphi(f) - \int f\psi \in L^*_{p,q}$, and by the above, $\zeta(f) = 0$ whenever f has compact support.[3] By cor. 1.iv, those f's are dense, so, by continuity of ζ, $\zeta(f) = 0\ \forall f \in L_{p,q}$: $\int f\psi = \varphi(f)$ on $L_{p,q}$. ■

Corollary 1. *(i)* $\|f\|_{p,q} = \lim_{K\in\mathcal{K}}\|f\mathbb{1}_K\|_{p,q}$
(ii) If $p < \infty$, $f = \lim_{K\in\mathcal{K}} f\mathbb{1}_K$ *in* $L_{p,q}$
(iii) For $K \in \mathcal{K}$, *the norm* $f \mapsto \|f\mathbb{1}_K\|_{p,q}$ *is equivalent to the norm* $f \mapsto \|f\mathbb{1}_K\|_q$
(iv) If $p, q < \infty$, *continuous functions with compact support are dense in* $L_{p,q}$

Proof. iii: By thm. 1.ii, choose for B a compact neighbourhood of K. Then, if $\mathsf{Supp}\, f \subseteq K$, $\|\mathbb{1}_{x+B} f\|_q$, $\|\mathbb{1}_{x+B} f\|_q$ equals $\|f\|_q$ on the compact neighbourhood of 0 $C \stackrel{\text{def}}{=} \cap_{x\in K}(x-B)$, is 0 outside $K-B$, and elsewhere lies in between. Thus if $p = \infty$ the result follows trivially, and else $(\lambda(C))^{1/p}\|f\|_q \leq \|f\|_{p,q} \leq (\lambda(K-B))^{1/p}\|f\|_q$.

i: follows from thm. 1.x.

ii: $|f| \geq \varepsilon$ is relatively compact by thm. 1.xii, and $f\mathbb{1}_{|f|\geq\varepsilon} \to f$ by thm. 1.xi.

iv: $f\mathbb{1}_{|f|\geq\varepsilon}$ can in turn be approximated by continuous functions with the same compact support, using iii. ■

Corollary 2. $\exists K\colon \|x \mapsto \|t \mapsto f(x-t)g(t)\|_q\|_p \leq K\|f\|_{1,\infty}\|g\|_{p,q}$.

Proof. The worst case is when $f \geq 0$, $g \geq 0$. Assume, adjusting K, that the norm used for f is as described in thm. 1.iv; the worst case is then when f is constant on each $j + B$, $j \in J$: $f = \sum_n f_n \mathbb{1}_{j_n+B}$ for some sequence j_n in J. Since the L_p norms are subadditive along sequences, the left hand member is $\leq \sum_n f_n \|x \mapsto \|t \mapsto \mathbb{1}_{j_n+B}(x-t)g(t)\|_q\|_p, = \|x \mapsto \|t \mapsto \mathbb{1}_{x-j_n-B}\, g\|_q\|_p \sum_n f_n$ because the norm is constant in n, the different functions of x differing only by their shift by j_n. Hence the result, since $\sum_n f_n = \|f\|_{1,\infty}$ and, using $j_n = 0$, the norm is $\leq \|g\|_{p,q}$. ■

Definition 2. $H \subseteq L_{p,q}$ is *tight* (or: (p,q)*-tight*) iff it is bounded and $\forall\varepsilon > 0$ $\exists K \in \mathcal{K}$ s.t. $\forall f \in H$, $\|f\mathbb{1}_{\complement K}\|_{p,q} \leq \varepsilon$.

Remark 2. Cor. 1.ii states that, for $p < \infty$, finite sets are tight.

Lemma 1. *If* $f_n \to f$ *locally in measure and* $|f_n| \leq g_n$ *where* g_n *is relatively compact in* $L_{p,q}$ *for* $p, q < \infty$, *then* $f_n \to f$ *in* $L_{p,q}$. *The same holds for* $q = \infty$ *if the* f_n *are equicontinuous, and then* $f_n \to f$ *in* C_p.

Proof. Since $p < \infty$ there is a σ-compact set carrying all g_n (cor. 1.ii). Extract thus a subsequence s.t. $f_n \to f$ a.e., and $g_n \to g$ in $L_{p,q}$, and s.t. further $\|g_n - g\|_{p,q}$ is summable. Then $h = g + \sum_n |g_n - g| \in L_{p,q}$, and $|f_n - f| \leq 2h\ \forall n$. Thus $f_n \to f$ in $L_{p,q}$ (dominated convergence), and f being independent of the subsequence, the same holds for the original sequence. If $q = \infty$, equicontinuity of the f_n implies

[2] One could as well deduce the existence of such χ_z from the converse in xiii.

[3] So, even if $p = \infty$, $\psi \in L_{p',q'}$ is the "σ-additive part" of $\varphi \in L^*_{p,q}$, and ζ its part at ∞; and there is an obvious counterpart to be shown for $q = \infty$.

first that their a.e. limit f is also continuous, and a pointwise limit, so the $h_n = \sup_{k>n}|f_k - f|$ are also equicontinuous, and decrease pointwise to 0, hence uniformly on compact sets. Thus $x \mapsto \sup 1\!\!1_{x+B} h_n \in L_p$ decreases pointwise to 0. ∎

Proposition 1. *(i) The $\sigma(L_{\infty,1}, L_{1,\infty})$ topology coincides on bounded sets with the topology of weak-convergence in L_1 on compact sets. (ii) For $p > 1$, a set is $\sigma(L_{p,1}, L_{p',\infty})$-relatively compact iff it is bounded and locally uniformly integrable. (iii) A set is $\sigma(L_{1,\infty}, L_{\infty,1})$-relatively compact iff it is tight. (iv) On $(1,\infty)$-tight sets, the Mackey topology $\tau(L_{1,\infty}, L_{\infty,1})$, the topology of local convergence in measure and the L_1 topology coincide, and the norm topology, the topology of uniform convergence on compact sets and the L_∞ topology also.*

Proof. i: the $L_{1,\infty}$-topology is that of uniform convergence on bounded subsets of $L_{\infty,1}$ (thm. 1.xiv) and the functions with compact support are dense.

ii: Weak-compactness implies boundedness; local uniform integrability is needed by the weak-compactness criterion in L_1 spaces on a finite measure space. Conversely, by the latter a locally uniformly integrable set is relatively weakly compact in each $L_1(K)$, $K \in \mathcal{K}$. By the Eberlein-Šmulian theorem [e.g. 1, 17.12 p. 159], we can also assume the set is a sequence f_n, and it suffices to show it has a cluster point.

If $p < \infty$, each f_n is carried by a σ-compact set, hence the sequence too; so one can extract a subsequence converging weakly in L_1 on each of those compact subsets K_k, assumed increasing w.l.o.g. For the limit f thus defined, and any $B \subseteq K_k$, $\|1\!\!1_B f\|_1 \le \liminf_n \|1\!\!1_B f_n\|_1$, so Fatou's lemma yields $\forall k, \|1\!\!1_{K_k} f\|_{p,1} \le \liminf_n \|1\!\!1_{K_k} f_n\|_{p,1}$, hence $\|f\|_{p,1} \le \liminf_n \|f_n\|_{p,1}$ (monotone convergence, thm. 1.x). Clearly any cluster point of the f_n in $L_1(K)$, $K \in \mathcal{K}$, must equal f. Since f_n is relatively weakly compact in $L_1(K)$, f_n converges to f weakly in $L_1(K)$ $\forall C$. So $f_n \to f$ on all functions with compact support in L_∞. Since those are dense in $L_{p',\infty}$ $(p > 1)$, and the sequence $f_n, \dots, f$ is bounded in $L_{p,1}$, $f_n \to f$ $\sigma(L_{p,1}, L_{p',\infty})$.

If $p = \infty$, take a *concassage* K_α: the set is a product of relatively weakly compact sets in $L_1(K_\alpha)$. So any ultrafilter $\mathcal{U}$ on the set has a weak limit in $\prod_\alpha L_1(K_\alpha)$, which can be identified with a measurable function f on G. Further, the set being weakly relatively compact in $L_1(K)$ $\forall K \in \mathcal{K}$, $\mathcal{U}$ has a weak limit in $L_1(K)$, which can only be f: $\mathcal{U}$ converges to f on all bounded measurable functions with compact support. The boundedness condition implies then that f satisfies the same bound, thus is in $L_{\infty,1}$. The functions in L_∞ with compact support being dense in $L_{1,\infty}$, boundedness of the set ensures $\mathcal{U}$ still converges to f in duality with $L_{1,\infty}$.

iii: A tight set H can be assumed convex, and because of tightness everything happens in a σ-compact subset of G. Therefore each point in the (compact) $\sigma(L_\infty, L_1)$-closure $\bar{H}$ of H is the limit of a sequence in H; by convexity, one can choose this sequence to be $\tau(L_\infty, L_1)$ convergent, i.e., convergent in measure on compact subsets, and hence, extracting a subsequence, it is the limit of an a.e. convergent sequence in H. Thus, by Fatou, $\bar{H}$ satisfies the same tightness condition as H itself, so we can assume $H = \bar{H}$, i.e., that H is $\sigma(L_\infty, L_1)$-compact. Again by the σ-compactness, any point in the $\sigma(L_\infty, L_1)$-closure $\bar{A}$ of $A \subseteq H$ is the limit of a $\sigma(L_\infty, L_1)$-convergent sequence in A; tightness implies then this sequence is $\sigma(L_{1,\infty}, L_{\infty,1})$-convergent: $\sigma(L_\infty, L_1)$-closed subsets of H are still $\sigma(L_{1,\infty}, L_{\infty,1})$-closed, so the 2 topologies coincide on H, which is thus $\sigma(L_{1,\infty}, L_{\infty,1})$-compact.

Conversely (*"méthode de la bosse glissante"*), assume H is $\sigma(L_{1,\infty}, L_{\infty,1})$-compact and not tight. By weak*-compactness, H is bounded; so $\exists \varepsilon > 0 \colon \forall K \in \mathcal{K}\, \exists f_K \in H$ s.t. $\|f_K 1\!\!1_{\complement K}\|_{1,\infty} > \varepsilon$. Define the norms by a compact neighbourhood B of 0, then inductively $C_0 = \emptyset$, $f_n = f_{C_n}$, $K_n \in \mathcal{K}$ s.t. $K_n \supseteq C_n$ and $\forall i \le n$ $\|f_i 1\!\!1_{\complement K_n}\|_{1,\infty} < n^{-1}$ (rem. 2), then $C_{n+1} \in \mathcal{K}$ s.t. $K_n + (B - B) \subseteq C_{n+1}$. So $C_n \subseteq K_n \subseteq C_{n+1}$, $(K_n - B) \cap (\complement C_{n+1} - B) = \emptyset$, and $\|f_n 1\!\!1_{\complement C_n}\|_{1,\infty} > \varepsilon$, $\|f_i 1\!\!1_{\complement K_n}\|_{1,\infty} < n^{-1}$ $\forall i \le n$. Since the f_i live thus on the σ-compact subset $\bigcup_n C_n$ of G, extract from them a subsequence converging $\sigma(L_\infty, L_1)$, say to f, renaming it as f_n itself: all above properties are preserved. By compactness of H, $f_n \to f$ in H; and clearly f is carried by $\bigcup_n C_n$. Let $h(x) = \operatorname{ess\,sup}(\mathbb{1}_{x+B}|f|)$: $h \in L_1$, and is carried by $\bigcup_n C_n - B$, $= \bigcup_n C_n$ because $K_n - B \subseteq C_{n+1}$. Thus $\exists n$ s.t. $\int_{\complement C_n} |h| < \frac{\varepsilon}{2}$, hence $\|f \mathbb{1}_{\complement C_n}\|_{1,\infty} < \frac{\varepsilon}{2}$. Thus $f_n - f$ is not tight, and converges $\sigma(L_{1,\infty}, L_{\infty,1})$ to 0. We can thus assume that our original set H was $\{f_n - f \mid n \geq 0\} \cup \{0\}$, and that our above construction was carried out with that H: now $f_n \to 0$ $\sigma(L_{1,\infty}, L_{\infty,1})$ in addition. Let now g_n have support in $K_n \setminus C_n$ and be s.t. $\|g_n\|_{\infty,1} = 1$, $\int f_n g_n \geq \delta(\varepsilon - n^{-1})$, where δ is the constant stemming from the norm-equivalence in thm. 1.xiii. Construct then inductively a subsequence n_i as follows: $n_0 = 0$; assume $n_0 \dots n_i$ known; since $f_n \to 0$, $\exists n_{i+1} > n_i$ s.t. $\forall n \geq n_{i+1}, \forall k \leq n_i, |\int f_n g_k| < (i+1)^{-2}$. Renaming the subsequence as just f_i, g_i etc., all our previous properties still hold, and now in addition $\forall k < n, |\int f_n g_k| < n^{-2}$. Then $g = \sum_n g_n$ is s.t. $\|g\|_{\infty,1} = 1$, because $(K_n - B) \cap (\complement C_{n+1} - B) = \emptyset$. And $\int f_n g \geq \delta(\varepsilon - n^{-1}) - 2n^{-1}$, which is $> \frac{1}{2}\delta\varepsilon > 0$ for n sufficiently large, contradicting that $f_n \to 0$ $\sigma(L_{1,\infty}, L_{\infty,1})$.

iv: Since the unit ball of L_∞ is $\sigma(L_{\infty,1}, L_{1,\infty})$-compact, the Mackey topology is stronger than that induced by L_1. Conversely, if a net is tight and converges locally in measure, consider a compact set C corresponding to ε: since a weakly compact set is norm-bounded, say by K, the total mass of the net outside C contributes at most $K\varepsilon$ to the result, and on C, the boundedness implies boundedness of the net in L_∞, hence its Mackey convergence is classical.

The last statement is obvious. ■

2. Convolution

Definition 3. Take 2 Radon measures μ and ν s.t. $|\mu \times \nu|\big(\{x, y \mid x + y \in K\}\big) < \infty$ $\forall K \in \mathcal{K}$. Define their *convolution* as the measure $\mu \star \nu$: $f \mapsto \int f(s+t)\mu(ds)\nu(dt)$ for every continuous function f with compact support.

Remark 3. For positive measures, if either $(\mu \star \nu) \star \rho$ or $\mu \star (\nu \star \rho)$ is defined, so is the other and both are equal.

Remark 4. The convolution is defined on M, and turns M into a commutative Banach algebra. L_1 is identified with a subspace of M.

Lemma 2. *If f is Haar-measurable (resp., -integrable), $f(t-s)$ is $\lambda(dt) \times \mu(ds)$-measurable (resp., -integrable) $\forall \mu \in M$, for the completion of the Borel extension of $\lambda \times \mu$ on $G \times G$, and its equivalence class depends only on that of f.*

Proof. Consider first a Borel f, and use Fubini's theorem, and that any Haar locally negligible set contained in a Borel set of measure 0. ■

Definition 4. For $f \geq 0$ Haar-measurable and $\mu \in M_+$ the convolution $\mu \star f$ is the equivalence class of $t \mapsto \int \tilde{f}(t-s)\mu(ds)$, for any $\tilde{f} \in f$. For f Haar-measurable and $\mu \in M$, $\mu \star f \overset{\text{def}}{=} \mu_+ \star f_+ - \mu_- \star f_+ - \mu_+ \star f_- + \mu_- \star f_-$ if a.e. well-defined.

Remark 5. For $f \geq 0$ Lebesgue-measurable on $\mathbb{R}$, $f \notin L_{\infty,1} \Rightarrow \exists \mu$: $\mu \star f = \infty$ a.e.

Proof. $\exists x_n$: $\int_{x_n-1}^{x_n} f_t dt \geq 2^n$. So for μ_1 Lebesgue measure on $[-1, 1]$ and $f_1 \overset{\text{def}}{=} \mu_1 \star f$, $f_1(x) \geq 2^n$ on $[x_n - 1, x_n]$; thus, with $\mu_2 = \sum n^{-2}\epsilon_{-x_n}$, $f_2 \overset{\text{def}}{=} \mu_2 \star f_1 = \infty$ on $[-1, 0]$. Convolution with a μ_3 with full support is then ∞ a.e. ■

Theorem 2. *(i) For $(f, \mu) \in L_{p,q} \times M$, $\mu \star (f d\lambda)$ and $(\mu \star f)d\lambda$ are well-defined and equal. (ii) Convolution defines injective, norm 1 algebra homomorphisms of M into the Banach algebra of endomorphisms of every $L_{p,q}$ and C_p space. (iii) For $p, q < \infty$, and the $\sigma(M, C_\infty)$ topology on M, $(\mu, f) \mapsto \mu \star f$ from $M \times L_{p,q}$ to $L_{p,q}$ or from $M \times C_p$ to C_p is jointly continuous when restricted to compact subsets of M.*

Proof. $\int_{z+B}(|\mu| \star |\tilde{f}|)(t)dt = \int\bigl[\int_{z-s+B}|\tilde{f}(t)|dt\bigr]|\mu|(ds)$ (lemma 2, Fubini), so $\|\mu \star f\|_{\infty,1} \leq \|\mu\|\|f\|_{\infty,1}$: $\mu \star f$ is well-defined on $L_{\infty,1}$, thus (thm. 1.vii and viii) on all $L_{p,q}$. $\mu \star (f d\lambda) = (\mu \star f)d\lambda$ follows then too, hence i.

We show now, for continuous f with compact support, that $\|\mu\star f\|_{p,q} \leq \|\mu\|\|f\|_{p,q}$ and $\mu_\alpha\star f\to 0$ in C_1 (α in a directed set A) if $\mu_\alpha\to 0$ in $\sigma(M, C_\infty)$ is relatively compact.

f is uniformly continuous, so it and its translates are uniformly equicontinuous, so the $\mu_\alpha \star f$ are uniformly equicontinuous.[4] Since $\mu_\alpha \to \mu$ implies their pointwise convergence, that convergence is uniform on compact sets.

If all μ_α are carried by a fixed compact set K, all $\mu_\alpha \star f$ vanish outside the compact set $K + \mathsf{Supp}\, f$, so this uniform convergence implies $\mu_\alpha \star f \to \mu \star f$ in all $\|\cdot\|_{p,q}$ (i.e., in C_1). So, for μ with compact support K, taking the $\mu_\alpha \to \mu$ with finite support $\subseteq K$ and s.t. $\|\mu_\alpha\| \leq \|\mu\|$, shift-invariance (for point masses) and convexity of the norm (thm. 1.i) imply $\|\mu_\alpha\star f\|_{p,q} \leq \|\mu_\alpha\|\|f\|_{p,q}$, and thus, by the convergence in all $\|\cdot\|_{p,q}$, $\|\mu \star f\|_{p,q} \leq \|\mu\|\|f\|_{p,q}$ for all continuous f with compact support.

Any $\mu \in M$ is the sum of a norm-summable series of μ's with compact support, and the corresponding series of convolutions with f is then also norm-summable, hence convergent, $L_{p,q}$ being a Banach space. By thm. 1.vii and 1.viii, a fortiori the series converges to the same limit in $L_{\infty,1}$. But since (cf. supra) $\|\mu \star f\|_{\infty,1} \leq \|\mu\|\|f\|_{\infty,1}$ on $M \times L_{\infty,1}$, the limit there can only be $\mu \star f$: the series converges in $L_{p,q}$ to $\mu \star f$, hence the inequality goes to the limit: $\forall f$ continuous with compact support, $\forall \mu \in M$, $\|\mu \star f\|_{p,q} \leq \|\mu\|\|f\|_{p,q}$.

For general μ_α, relative compactness implies tightness: $\forall\varepsilon > 0\ \exists K \in \mathcal{K}$: $\forall\alpha$, $|\mu_\alpha|(\complement K) \leq \varepsilon$. Let then $\tilde{\mu}_\alpha = \mu_{\alpha|K}$, and, for an ultrafilter $\mathcal{U}$ refining the natural filter on A, $\tilde{\mu} = \mathsf{lim}_{\mathcal{U}}\, \tilde{\mu}_\alpha$. By the above, $\mathsf{lim}_{\mathcal{U}}\, \tilde{\mu}_\alpha \star f = \tilde{\mu} \star f$ in all $\|\cdot\|_{p,q}$. Since $\|\tilde{\mu}_\alpha - \mu_\alpha\| \leq \varepsilon$ and hence $\|\tilde{\mu} - \mu\| \leq \varepsilon$, we get that $\|\tilde{\mu}_\alpha \star f - \mu_\alpha \star f\|_{p,q}$ and $\|\tilde{\mu} \star f - \mu \star f\|_{p,q}$ are $\leq \varepsilon\|f\|_{p,q}$, so $\mathsf{lim}_{\mathcal{U}}\|\mu_\alpha \star f - \mu \star f\|_{p,q} \leq 2\varepsilon\|f\|_{p,q}$; ε being arbitrary, this limit is 0; finally, $\mathcal{U}$ being arbitrary, $\mu_\alpha \star f \to \mu \star f$ in $\|\cdot\|_{p,q}$.

Consider next the inequality $\|\mu \star f\|_{p,q} \leq \|\mu\|\|f\|_{p,q}$ for a general $f \in L_{p,q}$.

For $p, q < \infty$, there exists by cor. 1.iv a sequence f_n of continuous functions with compact support converging to f. The inequality implies $\mu\star f_n$ is a Cauchy sequence in $L_{p,q}$, thus convergent there, say to g. By thm. 1.vii and 1.viii, a fortiori $f_n \to f$ and $\mu\star f_n \to g$ in $L_{\infty,1}$. But since (cf. supra) $\|\mu\star f\|_{\infty,1} \leq \|\mu\|\|f\|_{\infty,1}$ on $M \times L_{\infty,1}$, $\mu \star (f_n - f) \to 0$ in $L_{\infty,1}$: $g = \mu \star f$, so $\mu \star f_n \to \mu \star f$ in $L_{p,q}$. Hence $\|\mu \star f\|_{p,q} = \mathsf{lim}\|\mu \star f_n\|_{p,q} \leq \|\mu\|\, \mathsf{lim}\|f_n\|_{p,q} = \|\mu\|\|f\|_{p,q}$: the inequality holds $\forall f \in L_{p,q}$.

For $L_{p,\infty}$, normalise Haar measure s.t. $\lambda(B) = 1$: this does not affect the inequality to be proved, since $\|\cdot\|_{p,q}$ gets multiplied on both sides by the same constant. Then by thm. 1.vii $L_{p,\infty} \subset L_{p,q}$ with $\|\cdot\|_{p,\infty} \geq \|\cdot\|_{p,q}$, so $\|\mu\|\|f\|_{p,\infty} \geq \|\mu\|\|f\|_{p,q} \geq \|\mu \star f\|_{p,q}$. But $\|f\|_n \nearrow \|f\|_\infty$ on a probability space yields by monotone convergence that $\|g\|_{p,n} \nearrow \|g\|_{p,\infty}$, hence the inequality.

For $L_{\infty,q}$: by thm. 1.v $\|f\|_{\infty,q} = \mathsf{sup}_x\|\mathbb{1}_{x+B}f\|_q$, so by the duality of L_p spaces $\|f\|_{\infty,q} = \mathsf{sup}\{\int fg \mid \|g\|_{q'} \leq 1,\, g = 0 \text{ outside some } x+B\}$, $\frac{1}{q'} = 1-\frac{1}{q}$, and g can further be required to be bounded. Thus it suffices to show that $\int(\mu\star f)g \leq \|\mu\|\|f\|_{\infty,q}$ for any such g. The integral equals $\iint f(t-s)g(t)\lambda(dt)\mu(ds)$; indeed, Fubini applies by lemma 2 since g is bounded and has compact support. By Hölder, the inner integral is $\leq \|\mathbb{1}_{\mathsf{Supp}(g)}f\|_q \leq \|f\|_{\infty,q}$, hence the claim: $\|\mu\star f\|_{p,q} \leq \|\mu\|\|f\|_{p,q} \forall f, p, q$.

Consider now iii in general.

Since $p, q < \infty$, any $f \in L_{p,q}$ is by cor. 1.iv the limit of a sequence f_n of continuous functions with compact support. The $\phi_{f_n} = \mu \mapsto \mu \star f_n$ are $\sigma(M, C_\infty)$-continuous

[4]The modulus of uniform continuity of f (i.e., the map ϕ from neighbourhoods V of 0 to $\mathbb{R}$ s.t. $\phi(V) = \mathsf{sup}_{x-y\in V}|f_x - f_y|$) is, by integration, when multiplied by $\mathsf{sup}\|\mu_\alpha\|$ ($< \infty$ by relative compactness), $\geq$ that of the $\mu_\alpha \star f$.

on compact sets, as seen above, and, by the inequality, converge uniformly on compact sets to ϕ_f, which is thus $\sigma(M, C_\infty)$-continuous on compact sets K, $\forall f \in L_{p,q}$. Joint continuity with $\mu \in K$ follows then from $\|\phi_f - \phi_g\|_{p,q} \le \mathsf{sup}_{\mu \in K}\|\mu\| \|f-g\|_{p,q}$.

Remain thus only the following five points.

μ is an endomorphism of any C_p: $\mu \star f$ is continuous when f is so, as seen above, and μ and f have compact support; this holds thus still for an arbitrary continuous f, since the value of $\mu \star f$ in the neighbourhood of any given point depends only on the values of f on some compact set, so f can be modified outside that such as to have compact support. Hence, for $f \in C_\infty$, $\mu \star f \in C_\infty$ too, $\forall \mu \in M$, μ being the sum of a norm-summable series of measures with compact support. Since by definition $C_p = C_\infty \cap L_{p,\infty}$ (with $\|\cdot\|_{p,\infty}$), μ acts with norm $\|\mu\|$ on all C_p too.

The joint continuity property holds then on C_p ($p < \infty$) too, as above, continuous functions with compact support being dense in C_p.

Observe that the definition $(\mu \star \nu)(f) = \int f(s+t)\mu(ds)\nu(dt)$ extends in the usual way from continuous functions with compact support to all bounded Borel functions. This implies $\mu \star (\nu \star f) = (\mu \star \nu) \star f$ for all bounded Borel f; equality also holds for all negligible f by definition (def. 4), hence for all bounded Haar-measurable f. Thus, if $\mu, \nu \ge 0$, it holds for all Haar-measurable $f \ge 0$; hence for all $f \in L_{p,q}$ and all $\mu, \nu \in M$. So, the homomorphisms are algebra-homomorphisms.

Since $\epsilon_0 \in M$, they cannot have norm < 1.

As to their injectivity, suffices to prove it on the smallest space, C_1 by thm. 1.vii and viii. Take $\mu \in M$ s.t. $\mu \star f = 0 \, \forall f \in C_1$. C_1 being dense in L_1, the same holds for $f \in L_1$. Let f_V be the density of Haar measure normalised on compact neighbourhoods V of 0; then for continuous functions g with compact support $\int g(t)(\mu \star f_V)(t)\lambda(dt) = \int f_V(x)g(x+s)\lambda(dx)\mu(ds)$ converges to $\int g(s)\mu(ds)$ when V decreases to $\{0\}$, by the uniform continuity of g. Since $\mu \star f_V = 0$, we conclude that $\mu(g) = 0$ for all continuous g with compact support: $\mu = 0$. ■

Remark 6. The continuity property does not hold on $L_{\infty,q}$: e.g. for $f(x) = \mathsf{sin}(x^2)$, $\|(\epsilon_{\frac{1}{n}} - \epsilon_0) \star f\|_{\infty,q} \to 0$ is false, along any subsequence and for any q, though $f \in C_\infty \subseteq \cap_q L_{\infty,q}$, and though continuity holds on all C_p ($p < \infty$).

For $L_{p,\infty}$: $\|(\epsilon_{\frac{1}{n}} - \epsilon_0) \star \mathbb{1}_{[0,1]}\|_{p,\infty} = (2 + \frac{1}{n})^{\frac{1}{p}} \ge 1 \; \forall n, \forall p$ (using $B = [0,1]$).

By thm. 2.iii, A. Weil's compactness criterion [3, p. 53] still holds (same proof):

Corollary 3. *For $p, q < \infty$, $K \subseteq L_{p,q}$ is relatively compact iff it is (p,q)-tight and $\forall \varepsilon > 0$, $\exists V$ neighbourhood of 0 in G: $\forall x \in V, \forall f \in K, \|\$_x f - f\|_{p,q} < \varepsilon$. The same holds for C_p ($p < \infty$), using $q = \infty$.*

Proposition 2. *(i) $\exists K\colon \forall p_1, p_2, q \ge 1$ s.t. $p^{-1} \stackrel{\text{def}}{=} p_1^{-1} + p_2^{-1} - 1 \ge 0$, if $\|f\|_{p_1,q}\|g\|_{p_2,q'} < \infty$, where $q'^{-1} = 1 - q^{-1}$, then $f \star g$ is everywhere well-defined, and $\in C_p$, and $\|f \star g\|_{C_p} \le K \|f\|_{p_1,q}\|g\|_{p_2,q'}$.*

(ii) If $p_1, q < \infty$, and $E \subseteq L_{p_1,q}$ is relatively compact (or just 'equicontinuous': $x \mapsto \$_x f$ from G to $L_{p_1,q}$ is equicontinuous (at 0) for $f \in E$) and $B \subseteq L_{p_2,q'}$ is bounded, then convolution maps $E \times B$ to a bounded uniformly equicontinuous set H, with modulus of continuity $\mathsf{sup}_{h \in H} \mathsf{sup}_x |h(x+z) - h(x)| \le (K \,\mathsf{sup}_{g \in B}\|g\|_{p_2,q'}) \,\mathsf{sup}_{f \in E}\|\$_z f - f\|_{p_1,q}$.

(iii) If $q = \infty$, point ii still holds, replacing $L_{p_1,q}$ by C_{p_1}.

Proof. The result is clear if either $\|f\|_{p_1,q}$ or $\|g\|_{p_2,q'} = 0$. Else both are finite. Increasing p_1 to $p_1^{-1} = 1 - p_2^{-1}$ preserves the finiteness, by thm. 1.viii, so, by thm. 1.xiii, $f \star g$ is everywhere well-defined, $f(t-x)g(x)$ being integrable in x $\forall t$.

We first prove the inequality sub i, interpreting $\|h\|_{C_p}$ as $\|x \mapsto \mathsf{sup}_{y \in x+B}|h(y)|\|_p$.

Let $F_z = \|\mathbb{1}_{z+B} f\|_q$, $G_z = \|\mathbb{1}_{z+B} g\|_{q'}$ and, with $h = f \star g$, $H_z = \|\mathbb{1}_{z+B} h\|_\infty$. Since $f(t-x)g(x)$ is integrable, by Fubini $h(t) = \frac{1}{\lambda(B)} \iint \mathbb{1}_{z+B}(x) f(t-x) g(x) dx dz$, so, by Hölder, $|h(t)| \leq \frac{1}{\lambda(B)} \int \|\mathbb{1}_{t-z-B} f\|_q G_z dz$, and thus $H_y = \mathsf{sup}_{t\in y+B}|h(t)| \leq \frac{1}{\lambda(B)} \int \|\mathbb{1}_{y-z+B-B} f\|_q G_z dz$. Since B is relatively compact with non-empty interior, there is a finite set $I \subseteq G$ s.t. $I + B \supseteq B - B$, thus $H_z \leq \frac{1}{\lambda(B)} \sum_{i\in I} \int F_{z+i-t} G_t dt = \frac{1}{\lambda(B)} \sum_{i\in I} (F \star G)(z+i)$, hence, by Young's inequality [4], and Minkowski's, $\|H\|_p \leq K\|F\|_{p_1}\|G\|_{p_2}$ with $K = \frac{\#I}{\lambda(B)}$, which is the desired inequality.

In particular, the inequality sub i holds with the sup norm on the left-hand side (thm. 1.viii). This implies then $f \star g$ is continuous if either $p_1, q < \infty$ or $p_2, q' < \infty$: e.g., in the first case, since $\$_z h - h = (\$_z f - f) \star g$, $\mathsf{sup}_x|h(x-z) - h(x)| \leq K\|\$_z f - f\|_{p_1,q}\|g\|_{p_2,q'}$, and by thm. 2.iii $z \mapsto \$_z f \colon G \to L_{p_1,q}$ is continuous. If neither $p_1, q < \infty$ nor $p_2, q' < \infty$, since we can't have $p_1 = p_2 = \infty$ nor $q = q' = \infty$, it must be that either $p_1 = p = \infty$, $p_2 = q = 1$ or $p_2 = p = \infty$, $p_1 = q' = 1$.

The two cases are dual. Assume the first: $\mathsf{sup}_x|(f \star g)(x)| \leq K\|f\|_{\infty,1}\|g\|_{1,\infty} < \infty$, and we want to show that $f \star g$ is continuous. By the inequality, and cor. 1.ii, approximating g by some g with compact support yields a uniform approximation for $f \star g$, thus preserving continuity: we can assume g has compact support, say K. Then continuity of $f \star g$ at x involves only values of f in a neighbourhood of $x - K$: f too can be assumed to have compact support. Then, by cor. 1.iii, $f \in L_1$ and $g \in L_\infty$, case ($p_1 = p_2' = q = 1$, by thm. 1.vi) where continuity is already proved.

ii: Relative compactness of E implies its equicontinuity by cor. 3; given this, the proof was done 2 paragraphs above. And iii is the same, using thm. 2.iii for C_p. ■

Next corollary is needed for cor. 5.

Corollary 4. *Convolution with an element of L_∞ is a sequentially continuous map from L_1 $\sigma(L_1, L_\infty)$ to C_∞ with the topology of compact convergence.*

Proof. By prop. 2.i, $L_1 \times L_\infty$ is mapped to C_∞. Assume h_n is uniformly integrable in L_1, $f \in L_\infty$, and let us show that $h_n \star f$ are equicontinuous. The h_n can then be assumed uniformly bounded, and with support in a fixed compact set K, by a uniform approximation in L_1, resulting (same formula) in a uniform approximation in C_∞, preserving equicontinuity. f can also be taken as an indicator function, of a Borel set B, by linearity and uniform approximation; then, equicontinuity being a local property, since the h_n are carried by K, B can also be assumed relatively compact. Finally, the h_n being now uniformly bounded, one can approximate $\mathbb{1}_B$ in L_1 by a continuous function with compact support. Equicontinuity is obvious then. If now the h_n are weakly convergent to h, the $h_n \star f$ converge pointwise, by definition of $\sigma(L_1, L_\infty)$, and hence uniformly on compact sets by equicontinuity. ■

Corollary 5. *Convolution with an element of $L_{1,\infty}$ is a sequentially continuous map from L_1 $\sigma(L_1, L_\infty)$ to C_1.*

Proof. Let $h \in L_{1,\infty}$; by prop. 2.i ($p_1 = p_2 = q = 1$) the map from L_1 to C_1 is continuous, hence weakly continuous. Thus if $f_n \to 0$ $\sigma(L_1, L_\infty)$, $h \star f_n \to 0$ weakly in C_1, and by cor. 4, the $h \star f_n$ are equicontinuous, so the $h \star f_n$ converge uniformly to 0 on compact sets. For a compact set K, $J_n^K \stackrel{\text{def}}{=} \|\mathbb{1}_{\complement K}(h \star f_n)\|_{1,\infty} \leq \int_{\complement K} \mathsf{sup}_{t\in x+B} \int |h(t-z)||f_n(z)| dz dx$; so, with $H_x = \mathsf{sup}_{t\in x+B}|h(t)|$, $H \in L_{1,\infty}$ (thm. 1.ii) and $J_n^K \leq \int_{\complement K} H(x-z)|f_n(z)| dz dx = \int G_K(z)|f_n(z)| dz$, where $G_K(z) = \int_{\complement K - z} H(y) dy$. There is a K_σ subgroup G_0 outside of which H and all f_n are negligible. Then, for $K_n \nearrow G_0$, the G_{K_n} are uniformly bounded and decrease pointwise to 0 on G_0, hence, by the weak compactness of the f_n, $\mathsf{lim}_{K\uparrow} \mathsf{sup}_n J_n^K = 0$. Together with the uniform convergence of the $h \star f_n$ to 0 on compact sets, this implies their C_1 convergence to 0. ■

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