Demand for Cash with Intra-Period Endogenous Consumption

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Abstract

We study the demand for money when agents can optimally choose mean rates of consumption and cash holdings over a period. Consistent with empirical evidence, we find that agents do not smooth intra-period consumption. Instead, their rate of consumption is positively correlated with their cash position. This positive correlation depends on the volatility of the consumption process. When volatility is very low or very high, agents choose to consume at a relatively high rate immediately after a cash withdrawal, drawing down quite rapidly their cash balances. Later in the period, their rate of consumption and cash depletion is more restrained. This sizeable deviation from consumption smoothing is much less pronounced when volatility is moderate.

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I. Introduction

To date, the literature on the demand for money has taken consumption as exogenous. Given the consumption process, which can be deterministic or stochastic, agents choose how much cash to withdraw at the start of a cycle. Cash holdings decline between withdrawals in line with the consumption process. Original contributions to this literature were made by Baumol (1952), Tobin (1956), and Miller and Orr (1966). More recent contributions include Frenkel and Jovanovic (1980), Bar-Ilan (1990), Bar-Ilan, Perry and Stadje (2004), Baccarin (2009), and Alvarez and Lippi (2009, 2010).

The assumption of exogenous consumption actually involves two separate assumptions. First, total consumption expenditures within the period, and consequently total money depleted during the period, are given. Second, the mean rate of consumption and cash depletion throughout the period is constant.

We retain the assumption of a given total consumption within the period as in the standard model of money demand. However, we extend the literature on money demand by relaxing the assumption of a constant consumption rate during the period. Instead, agents choose their mean rates of consumption and cash depletion as a function of their cash position. This causality from cash position to consumption reverses the traditional causality from consumption to cash that has characterized the literature on the demand for money. The extension permits the decision about money holdings and consumption to be made jointly.

Typically in the money demand literature, the consumption path over a period is characterized in continuous time as a Brownian motion (BM).[1] The parameters of the BM, namely the (negative) rate of drift and the instantaneous standard deviation of the process, are exogenous and fully characterize the path of consumption paid with cash and, correspondingly, the path of cash depletion.[2] Given the fixed parameters of the BM, as well as the opportunity cost of holding cash and the cost of restocking depleted cash balances, agents choose the optimal size of a cash withdrawal that minimizes total cost.

In our model, cash holdings and consumption are characterized by a BM, but agents can select the drift. The model is therefore a drift control model, where the drift is the mean rate of consumption and cash depletion.[3] For simplicity, we give agents just two opportunities to optimize their consumption rate during the period, first when they make their initial cash withdrawal and again when cash balances are reduced by half.[4] Hence at the start of a typical period, agents choose the optimal cash withdrawal, M*, the optimal rate of consumption for as long as money holdings do not reach M*/2, and the optimal rate of consumption afterwards. When cash balances hit their lower bound, assumed here to be zero, a cash withdrawal of size M* occurs and a new cycle starts. Agents therefore choose M* and two rates of consumption in order to minimize the

[1] In Bar-Ilan et al. (2004) and Alvarez and Lippi (2010), the underlying process is a combination of Brownian motion and Compound Poisson. This process is still exogenous and not under the control of agents.

[2] When applied to firms, the BM process represents the firms' net revenue in cash.

[3] Bar-Ilan, Marion and Perry (2007) illustrate another type of drift control model.

[4] Limiting the number of times agents observe their cash position and adjust rates of consumption is consistent with the literature on optimal inattention to the stock market (e.g. Abel and Eberly (2007)).

present value costs of withdrawing and holding money, subject to the constraint on total cash consumption over the period.[5]

Numerical results for the model solution show unambiguously that cash consumption depends positively on cash holdings. For a wide range of parameter values, and without exception, the rate of cash consumption immediately after a cash withdrawal is greater than the consumption rate chosen at M*/2. The difference in the two consumption rates is substantial for most parameter values. Consequently, when agents have the opportunity to choose their rates of consumption over the period, they do not smooth consumption but prefer to consume at a higher rate initially.

The degree of consumption smoothing within a period depends on the amount of volatility in the consumption process. When volatility is either very high or very low, agents adopt a rapid rate of consumption immediately after a cash withdrawal and then slow down that rate later in the period. For example, high volatility can cause agents to choose a rate of consumption in the first half of the period that is many times higher than the rate later in the period; low volatility leads consumers to make large purchases immediately upon a cash withdrawal. Consequently, the intra-period consumption path deviates substantially from consumption smoothing for extreme levels of volatility. For moderate volatility, agents get much closer to consumption smoothing within the period.

Our results are generally consistent with two strands of empirical literature. Evidence on the demand for cash indicates that cash consumption increases with the amount of cash held. Alvarez, Pawasutipaisit and Townsend (2010) uncovered evidence

[5] For the sake of completeness we later relax the constraint on total cash consumption during the period, although this case is less interesting.

of simultaneous large cash withdrawals and large cash expenditures using a data set of rural Thai households. Diary surveys on the use of cash versus other means of payment present cross-sectional evidence that consumers who use debit cards frequently hold less cash than others. This finding suggests a positive correlation between cash holdings and cash consumption. In addition, point estimates from money demand regressions generally reveal the income elasticity of cash to be less than 0.5, the value implied by the "square-root formula" derived in Baumol (1952).[6] Boeschoten (1992) suggests this lower income elasticity may reflect a deviation from the implicit assumption in the standard money demand model of equally spaced expenditures between cash withdrawals. Instead, "households spend a large part of money relatively soon after its acquisition" (p.61).

Similarly, data on household consumption indicate also that the path of within-the-month consumption may not be well represented by a single drift rate over the period. This is especially true for poor consumers who tend to make more cash purchases. For example, Stephens (2003) used the Consumer Expenditure Survey's Diary Survey to examine the response of consumption expenditures to the monthly arrival of Social Security checks. He found that in the first few days following receipt of a Social Security check there is an increase in the amount of spending across multiple categories of expenditure relative to the day before the check arrives. For poorer households, where Social Security represents a more significant portion of income, the spending increase at the beginning of the month is more pronounced. Mastrobuoni and Weinberg (2009) found evidence that Social Security recipients without savings do not smooth

[6] For example, see Stix (2003) on both the use of debit cards and on income elasticity.

consumption over the month. Instead, these individuals consume 25 percent fewer calories the week before receiving Social Security checks relative to the week afterward.[7]

We conclude this section by highlighting the dominant role of cash in household transactions, as implied by European survey and diary data.[8] For example, Mooslechner, Stix and Wagner (2006) found that cash payments accounted for 86 percent of all direct payment transactions by Austrian households in 2005 and for 70 percent of total payment value. A Bundesbank survey (2009) found that cash accounted for 82 percent of all direct payment transactions by German consumers in 2008 and for 58 percent in terms of value. Attanasio, Guiso and Jappelli (2002) found that currency is very important in the Italian payment system. Further, cash used by these European households for transactions was only a small part of total cash in circulation. The rest was hoarded, used in the shadow economy or held abroad.[9] Understanding more fully the management of cash holdings is therefore an important goal.

The rest of the paper is organized as follows. In Section II we present the model and its solution. In Section III we describe the results and offer some intuition. Section IV concludes. A detailed derivation is relegated to the Appendix.

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[7] Assuming hyperbolic discounting can also generate non-smooth consumption that decreases within the month.

[8] In the U.S., households use debit cards more frequently than either cash, credit cards or checks. (Federal Reserve Bank of Boston, 2010).

[9] Schneider, Buehn, and Montenegro (2010) estimated that the average value of the shadow--or cash-- economy was 34.5% of official GDP for 162 countries between 1999 and 2006/7. The cash economy was 38.7% of official GDP for a group of 98 developing countries and 18.7% for a group of 25 high-income countries.

II. The Model and its Solution

Let $M^0=\{M^0(t):t>0\}$ be a BM with drift μ_0, variance σ^2, and initial value $M^0(0)=M^*$. Define the stopping time T_0 as the first time when the drift is controlled to μ_1. It is defined as $T_0= \min\{t>0:M^0(t)\leq \frac{M^*}{2}\}$.

Given T_0, let $M^1=\{M^1(t):t>T_0\}$ be a BM with drift μ_1, variance σ^2, and initial value $M^1(T_0)=M^*/2$. The stopping time T_1 is the first time when the drift is controlled back to μ_0. It is given by $T_0+T_1= \min\{t>T_0:M^1(t)\leq 0\}$.

The cash level $\{M(t):t\geq 0\}$ is a regenerative process with cycle T_0+T_1 such that for $t\leq T_0+T_1$

$$M(t)=\begin{cases} M^0(t), t\leq T_0 \\ M^1(t), T_0<t\leq T_0+T_1 \end{cases}. \tag{1}$$

Note that $\{M(t):0\leq t\leq T_0\}$ is a BM with parameters (μ_0,σ^2) and that $\{M(t):T_0<t\leq T_0+T_1\}$ is a BM with parameters (μ_1,σ^2); also, $M(0)=M^*$ and $M(T_0)=M^*/2$.

Having described the dynamics of the drift control, we now model the costs associated with cash management. There are two types of costs, the cost of holding cash (foregone interest) and the cost of withdrawal.[10] The expected discounted cost of holding money is the foregone interest,

$$A_1 = rE_{M^*}\int_0^\infty e^{-rt}M(t)dt \tag{2}$$

[10] There is no cost of controlling the drift. A straightforward extension of this model is to introduce such a cost. Such a strategy will make endogenous the number of drifts between cash withdrawals.

where r is the interest rate and the expectations operator E_{M^*} is defined as

$$E_{M^*}(X) \equiv E(X \mid M(0) = M^*). \tag{3}$$

Since M(t) is a regenerative process, we can express A_1 in terms of a cycle. Let

$$\theta_0(r) = E_{M^*}(e^{-rT_0}), \tag{4}$$

and

$$\theta_1(r) = E_{\frac{M^*}{2}}(e^{-rT_1}) \tag{5}$$

In the Appendix we show that

$$A_1 = r\frac{E_{M^*}\int_0^{T_0} e^{-rt}M(t)dt + \theta_0(r)E_{\frac{M^*}{2}}\int_0^{T_1} e^{-rt}M(t)dt}{1-\theta_0(r)\theta_1(r)}. \tag{6}$$

There is also a cost k associated with each withdrawal. The expected discounted cost of all withdrawals is

$$A_2 = \frac{k\theta_0(r)\theta_1(r)}{1-\theta_0(r)\theta_1(r)}. \tag{7}$$

Total cost (TC), minimized by choosing the optimal values for M*, μ_0, and μ_1, is

$$TC = A_1 + A_2. \tag{8}$$

The solution requires solving for the two integrals in A_1 and for $\theta_i(r)$, i=0,1. Define $x_0(r)$ as

$$x_0(r) = \frac{\mu_0 + \sqrt{\mu_0{}^2 + 2r\sigma^2}}{\sigma^2} \tag{9}$$

Also note that

$$\theta_0(r) = e^{-x_0(r)M^*/2} \tag{10}$$

$$E_{M^*}\int_0^{T_0} e^{-rt}M(t)dt = \frac{r[M^* - \frac{M^*}{2}\theta_0(r)] + \mu_0(1-\theta_0(r))}{r^2}. \tag{11}$$

Similarly,

$$x_1(r) = \frac{\mu_1 + \sqrt{\mu_1{}^2 + 2r\sigma^2}}{\sigma^2} \tag{12}$$

$$\theta_1(r) = e^{-x_1(r)M^*/2} \tag{13}$$

$$E_{\frac{M^*}{2}}\int_0^{T_1} e^{-rt}M(t)dt = \frac{r\frac{M^*}{2} + \mu_1(1-\theta_1(r))}{r^2}. \tag{14}$$

The total cost of cash management is

$$TC = \frac{M^* + [\mu_0(1-\theta_0(r)) + \mu_1\theta_0(r)(1-\theta_1(r))]r^{-1} + k\theta_0(r)\theta_1(r)}{1-\theta_0(r)\theta_1(r)}. \tag{15}$$

Total cost in (15) is minimized by optimally choosing M* and μ_i, i=1,2, given the parameters (k, r, σ).

The steady-state density of cash balances (the density when $t \to \infty$) for a (μ, σ^2) BM , $\mu < 0$, with a trigger x that induces an impulse control of size (y - x) > 0 is[11]

$$\phi(M) = \begin{cases} (y-x)^{-1}(1-e^{-\pi(M-x)}), x \le M \le y \\ (y-x)^{-1}(e^{-\pi(M-y)} - e^{-\pi(M-x)}), y < M \end{cases} \tag{16}$$

where $\pi \equiv 2|\mu|/\sigma^2$. In our case the steady-state density is a weighted average of two densities. The first is a (μ_0, σ^2) BM with a trigger M*/2 and target M*, while the second is a (μ_1, σ^2) BM with a trigger 0 and target M*/2. The weight of the former is

$$\omega_0 = ET_0 /(ET_0 + ET_1) = |\mu_1|/(|\mu_0| + |\mu_1|), \tag{17}$$

while the weight of the latter is

$$\omega_1 = ET_1 /(ET_0 + ET_1) = |\mu_0|/(|\mu_0| + |\mu_1|). \tag{18}$$

ET_i, $i = 0,1$, denotes the expected first-passage time given by

$$ET_0 = \frac{M^* - M^*/2}{|\mu_0|} = \frac{M^*/2}{|\mu_0|},$$

$$ET_1 = \frac{M^*/2 - 0}{|\mu_1|} = \frac{M^*/2}{|\mu_1|}$$

[11] See Bar-Ilan (1990).

Equations (16)-(18) give the following steady-state density

$$\phi(M) = \begin{cases} (2/M^*)\omega_1(1-e^{-\pi_1 M}), 0 \le M < M^*/2 \\ (2/M^*)\{\omega_1(e^{-\pi_1(M-M^*/2)} - e^{-\pi_1 M}) + \omega_0(1-e^{-\pi_0(M-M^*/2)})\}, M^*/2 \le M < M^* \\ (2/M^*)\{\omega_1(e^{-\pi_1(M-M^*/2)} - e^{-\pi_1 M}) + \omega_0(e^{-\pi_0(M-M^*)} - e^{-\pi_0(M-M^*/2)})\}, M^* \le M \end{cases} \quad (19)$$

where

$$\pi_i \equiv 2|\mu_i|/\sigma^2, \text{i=0,1.} \quad (20)$$

This density yields the following steady-state mean value of cash, E(M), as a function of the mean values of the two BM processes, E_0(M) and E_1(M):

$$E(M) = \omega_0 E_0(M) + \omega_1 E_1(M), \quad (21)$$

where

$$E_0(M) = \frac{3}{4}M^* + \frac{\sigma^2}{2|\mu_0|} \quad (22)$$

$$E_1(M) = \frac{1}{4}M^* + \frac{\sigma^2}{2|\mu 1|}. \quad (23)$$

We first obtain a solution without constraining agents to have a given cash consumption over the period. We discuss the solution in the next section. We then consider the more interesting case where agents are constrained to have a given amount of cash consumption over the period. The total cash withdrawal within a given period of length T, say one month, is equal to a given level Y, which is the total cash consumption within this period.

The mean time between cash withdrawals is denoted as $E(T_0+T_1)$. Hence the constraint on total per-period consumption is,

$\frac{M^*}{E(T_0+T_1)} = \frac{Y}{T} \equiv y$, where y is a given parameter.[12] Since

$E(T_0+T_1) = \frac{M^*/2}{|\mu_0|} + \frac{M^*/2}{|\mu_1|}$, the constraint becomes

$$\frac{2}{\frac{1}{|\mu_0|}+\frac{1}{|\mu_1|}} = y\,. \tag{24}$$

The agent optimally chooses M* and the two drifts μ_0 and μ_1 by minimizing the cost function (15) subject to the constraint (24), given the parameters (k, r, y, σ).

III. Results

The model solutions are illustrated in Figures 1-4. These figures show how the solutions vary with volatility (σ) for the following parameter values: k=1, r=0.05, and y=5. Figure 1 illustrates the total cost of cash management. Figure 2 displays the two optimal rates of consumption (the drifts of the two BM processes). The optimal cash withdrawal at the start of the period is shown in Figure 3, while the steady-state mean value of cash holdings is given in Figure 4. The four figures contain not only the model solutions (denoted with a solid line), but the solutions for the standard money demand model with an exogenous consumption process (denoted with a dashed line). For the standard model, the BM process is characterized by one fixed drift, equal to 5, over the

[12] The expected number of cash withdrawals within a period T is $T/E(T_0+T_1)$, giving total cash withdrawn during time T as $TM^*/E(T_0+T_1)=Y$.

period. We can therefore assess what difference it makes when agents optimize their intra-period rates of consumption.

The solutions to the standard money demand model can be described briefly, as they are well known.[13] Higher volatility increases the chance that cash holdings will obtain extreme values away from the initial cash level M*. This higher probability is reflected in both higher holding costs of large amounts of cash and a higher restocking cost, as in Figure 1. To contain the latter cost, higher volatility raises the optimal cash withdrawal at the start of the period, M*, the only parameter of choice in this case (Figure 3). The average cash holding during the period rises with volatility as in Figure 4 both because M* rises and because of the direct effect of volatility on extreme levels of cash. Since the consumption process is exogenous, the drift is invariant to increases in volatility (Figure 2).

When agents optimally choose two rates of consumption over the period, the additional degrees of freedom allow them to reduce their total cost of cash management. This is reflected in Figure 1 as the solid line is always below the dashed one, regardless of the volatility level. When $\sigma = 1$ TC_{SM} = 13.92 (SM=standard model), which is 40 percent higher than the TC=9.94 of our model. The two costs are closest around $\sigma = 10$ but diverge again as volatility increases. For $\sigma = 80$, the cost difference is 9.7 percent (211.83.23 vs. 232.36).

The most interesting figure is Figure 2 which shows how agents are able to choose their intra-period rates of consumption. First, it is always the case that $|\mu_0| > |\mu_1|$. The rate of consumption immediately after a cash withdrawal, μ_0, is always

[13] See Frenkel and Jovanovic (1980).

greater in absolute value that the rate of consumption further into the period, μ_1. This outcome is true for a wide range of (k,r,y) parameter values. Given constraint (24) on the average rate of consumption, the two rates chosen by agents lie on either side of the exogenous rate faced in the standard money demand model. Provided the opportunity to choose their intra-period rates of consumption, agents deviate from consumption smoothing.

The difference between the two endogenously-determined consumption rates depends on volatility. When volatility is very low or very high, the initial drift μ_0 is highly negative. Agents choose a rapid rate of cash consumption immediately after a cash withdrawal. For our parameter values, the initial rate of consumption can be many times higher than the rate chosen for later in the period. When volatility lies between these extremes, the difference between consumption rates is much smaller. For some intermediate volatilities, consumption is quite smooth (consumption rates of 6.56 and later 4.04 for $\sigma = 14.5$).[14]

Two observations about these drift results are noteworthy. First, and most importantly, consumption is positively correlated with the level of cash holdings. The observation that the rate of consumption immediately after a cash withdrawal exceeds the rate of consumption later in the period is not due to impatience. It is the outcome of choosing an optimal consumption pattern that minimizes the cost of managing cash. Moreover, this outcome is consistent with survey and diary evidence on cash holdings and consumption patterns within a month. It is also consistent with point estimates from money demand regressions showing an income elasticity of cash under 0.5.

[14] The minimum difference between the two consumption rates falls with the average consumption rate y. The large difference for extreme levels of volatility persists.

Second, the optimal pattern of consumption over the period is consistent with the observation that agents sometimes choose to make large purchases immediately after a cash withdrawal. (See Alvarez et al, 2010.) Based on Figure 2, we see that agents are more inclined to make these large purchases when volatility is very low or very high. When there is little uncertainty, agents may be more willing to make large cash purchases at the beginning of the period because their future expenditures are predictable. When there is a great deal of uncertainty, like that arising from high and unstable inflation, agents may prefer to make large cash payments immediately after a cash withdrawal instead of waiting. During the German hyperinflation as well as contemporary high-inflation episodes in Argentina, Brazil, and elsewhere, it was common for households to purchase as many items as possible immediately after a wage payment.

In the absence of volatility, the initial drift μ_0 goes to negative infinity, the weight $\omega_0 = 0$ and E(M)=E_1(M).[15] In this case, half of the period's consumption (M*/2) takes place immediately upon a cash withdrawal; the other half is spent evenly during the month at a rate that is half of average period consumption. The cash spent immediately upon withdrawal, one-half of the withdrawal in our model, does not affect average cash holdings within the period. Alvarez and Lippi (2010) discuss a similar phenomenon and Alvarez et al. (2010) provide empirical support for this outcome.

Figure 3 shows that the optimal cash withdrawal is higher when agents choose their rates of intra-period consumption rather than face an exogenous consumption process. The reason is straightforward. With a higher initial consumption rate, agents

[15] In Figure 2, the range of this result is $\sigma < 6$. When the constraint on total cash consumption (y) is reduced, this range narrows, getting closer to $\sigma = 0$.

choose to withdraw more cash M* and deplete it more rapidly until M*/2. The larger is the deviation of the initial and average consumption rates, the larger the deviation of the size of withdrawal. Note also that unlike the standard model, M* is not monotonically increasing with volatility. When the initial drift rate drops dramatically (when σ is around 6-10), M* drops correspondingly.

Figure 4 shows average money holdings over the period for our model and for the standard one. When $\sigma < 5$, EM<EM_{SM}. When volatility increases, this inequality is reversed and the difference widens. This result is due to the lower consumption rate during the second part of the cycle, as average money holdings depends nonlinearly on the reciprocal of the consumption rate.[16] Note that despite making a larger cash withdrawal and holding more cash on average, the pattern of consumption spending ensures that the total cost of cash management is less for our model than in the case where agents have no discretion over their rates of consumption.

It might also be informative to note how the solutions differ when agents have the opportunity to optimize over their rates of consumption but do not face the constraint on total cash consumption over the period. We pursue two different approaches. The first involves choosing the size of withdrawal and one drift rate by minimizing the total cost (15) with respect to M* and $\mu = \mu_0 = \mu_1$. The second approach allows for the unconstrained choice of two drifts by minimizing (15) with respect to M*, μ_0 and μ_1. As expected, without a constraint on consumption, agents will choose low consumption rates, but these rates increase with volatility. Also, when agents can choose consumption

[16] There is also very narrow range around $\sigma = 6$ where EM drops with σ together with a large drop of M*.

rates and money holdings simultaneously as in the second version above, they choose to consume faster when they hold more cash.

IV. Conclusion

This paper studies the demand for cash when consumers have discretion over their rate of consumption during the period. A consequence of this extension is that agents always consume at a faster rate when they hold more cash. This deviation from consumption smoothing is larger when volatility takes extreme values. For intermediate values of volatility, consumption is smoother and cash holdings are closer to the values predicted by the standard model of money demand.

Although not pursued here, a straightforward application of this model is to evaluate numerically the income- and interest- rate elasticities of the demand for cash. Plotting E(M), the mean level of cash, as a function of the average consumption rate y and the interest rate r will yield these elasticities. The graph of E(M) as a function of r is potentially important as it allows for the computation of the welfare cost of inflation and a comparison with the welfare cost in the standard model.

An extension of the model would be to allow for $n>2$ drifts, where the drift changes whenever M(t) drops by M^*/n. If each control of drift incurs a (fixed or proportional) cost, then the number and timing of drift changes is chosen optimally to minimize the cost of control in addition to the other costs. Another possible extension would be to consider consumption processes other than BM such as Compound Poisson.

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Appendix

This appendix presents the derivation of the total expected cost of cash management.

Given that M(t) is a regenerative process with a cycle $T_0 + T_1$, we can write the total expected discounted cost of managing cash $TC(r)$ as

$$TC(r) = rE_{M*}\int_0^{T_0} e^{-rt}M(t)dt + r\theta_0(r)E_{\frac{M*}{2}}\int_0^{T_1} e^{-rt}M(t)dt + \theta_0(r)\theta_1(r)(k + TC(r)). \qquad \text{(A.1)}$$

Grouping the two terms $TC(r)$ on the left-hand side of (A.1) gives the expression for the total cost, equations (6)-(8). What remains on the right-hand-side of (A.1) is then the sum of the two costs associated with cash management—the holding cost and the withdrawal cost. These two costs are called A_1, equation (6), and A_2, equation (7).

To compute the functional forms of $\theta_i(r), E_z\int_0^{T_i} e^{-rt}M(t)dt$, $i = 0,1$, we generalize the technique used in Bar-Ilan et al. (2004) and Perry and Stadje (1999). The main tool of our analysis is a martingale N(t). It follows from Ito's Lemma (see chapter 5 of Chung and Williams (1990)) that if U is a BM with exponent $\varphi(\alpha) = (1/2)\sigma^2\alpha^2 - \mu\alpha$, $V = \{V(t): t \geq 0\}$ is an adapted process of bounded variation on finite intervals, and $W = \{W(t): t \geq 0\}$ satisfies $W(t) = U(t) + V(t)$, then

$$N(t) = \varphi(\alpha)\int_0^t e^{-\alpha W(s)}ds + e^{-\alpha W(0)} - e^{-\alpha W(t)} - \alpha\int_0^t e^{-\alpha W(s)}dV(s) \qquad \text{(A.2)}$$

is a martingale. We use this martingale as follows. Since $\{M(t): t \geq 0\}$ is a regenerative process with cycle $T_0 + T_1$, we divide the cycle into two parts and analyze each of them separately. The first part is $\{M(t): t \leq T_0\}$, which is a BM with $M(0) = M*$, $M(T_0) = M*/2$, drift $\mu_0 \in (-\infty, \infty)$ and variance $\sigma^2 > 0$. The second part is

$\{M(t): T_0 < t \le T_1\}$ which is BM with $M(T_0) = M^*/2,\ M(T_0+T_1) = 0,$ drift $\mu_1 \in (-\infty, \infty)$ and variance $\sigma^2 > 0$.

To use the martingale (A.2) on the first part of the cycle, set $\varphi(\alpha) = \varphi_0(\alpha) = (1/2)\sigma^2\alpha^2 - \mu_0\alpha$, $U(t) = M(t)$, $V(t) = (r/\alpha)t$, and $W(t) = M(t) + (r/\alpha)t$. Then

$$N_0(t) = \varphi_0(\alpha)\int_0^t e^{-\alpha M(s)-rs}ds + e^{-\alpha M^*} - e^{-\alpha M(t)-rt} - r\int_0^t e^{-\alpha M(s)-rs}ds \qquad \text{(A.3)}$$

is a martingale. By setting $E_{M^*}N_0(0) = E_{M^*}N_0(T_0)$, we obtain

$$\varphi_0(\alpha)E_{M^*}\int_0^{T_0} e^{-\alpha M(s)-rs}ds = -e^{-\alpha M^*} + E_{M^*}e^{-\alpha M(T_0)-rT_0} + rE_{M^*}\int_0^{T_0} e^{-\alpha M(s)-rs}ds\,. \qquad \text{(A.4)}$$

Rearranging terms in (A.4), using $M(T_0) = M^*/2$, yields

$$(\varphi_0(\alpha) - r)E_{M^*}\int_0^{T_0} e^{-\alpha M(s)-rs}ds = -e^{-\alpha M^*} + E_{M^*}e^{-\alpha M(T_0)-rT_0} = -e^{-\alpha M^*} + e^{-\alpha M^*/2}\theta_0(r) \qquad \text{(A.5)}$$

with $\theta_0(r)$ defined earlier as $\theta_0(r) = E_{M^*}(e^{-rT_0})$.

Let x_0 be the positive root of the quadratic equation $\varphi_0(\alpha) - r = (\sigma^2/2)\alpha^2 - \mu_0\alpha - r = 0$, so that

$$x_0(r) = \frac{\mu_0 + \sqrt{\mu_0^2 + 2r\sigma^2}}{\sigma^2}. \qquad \text{(A.6)}$$

Equation (A.6) is equation (9), section 2. Substituting $\alpha = x_0(r)$ into equation (A.5) makes the left-hand-side equal to zero. Equation (A.5) therefore yields the following equation for $\theta_0(r)$,

$$\theta_0(r) = e^{-x_0(r)M^*/2}. \qquad \text{(A.7)}$$

Equation (A.7) is equation (10).

Now substitute equation (A.7) into (A.5), divide both sides by $r-\varphi_0(\alpha)$, take the derivative with respect to α and set $\alpha = 0$. This yields

$$E_{M^*}\int_0^{T_0} e^{-rt}M(t)dt = \frac{r[M^* - \frac{M^*}{2}\theta_0(r)] + \mu_0(1-\theta_0(r))}{r^2}, \tag{A.8}$$

which is equation (11).

The solution technique for the second part of the cycle is similar and yields equations (12)-(14).

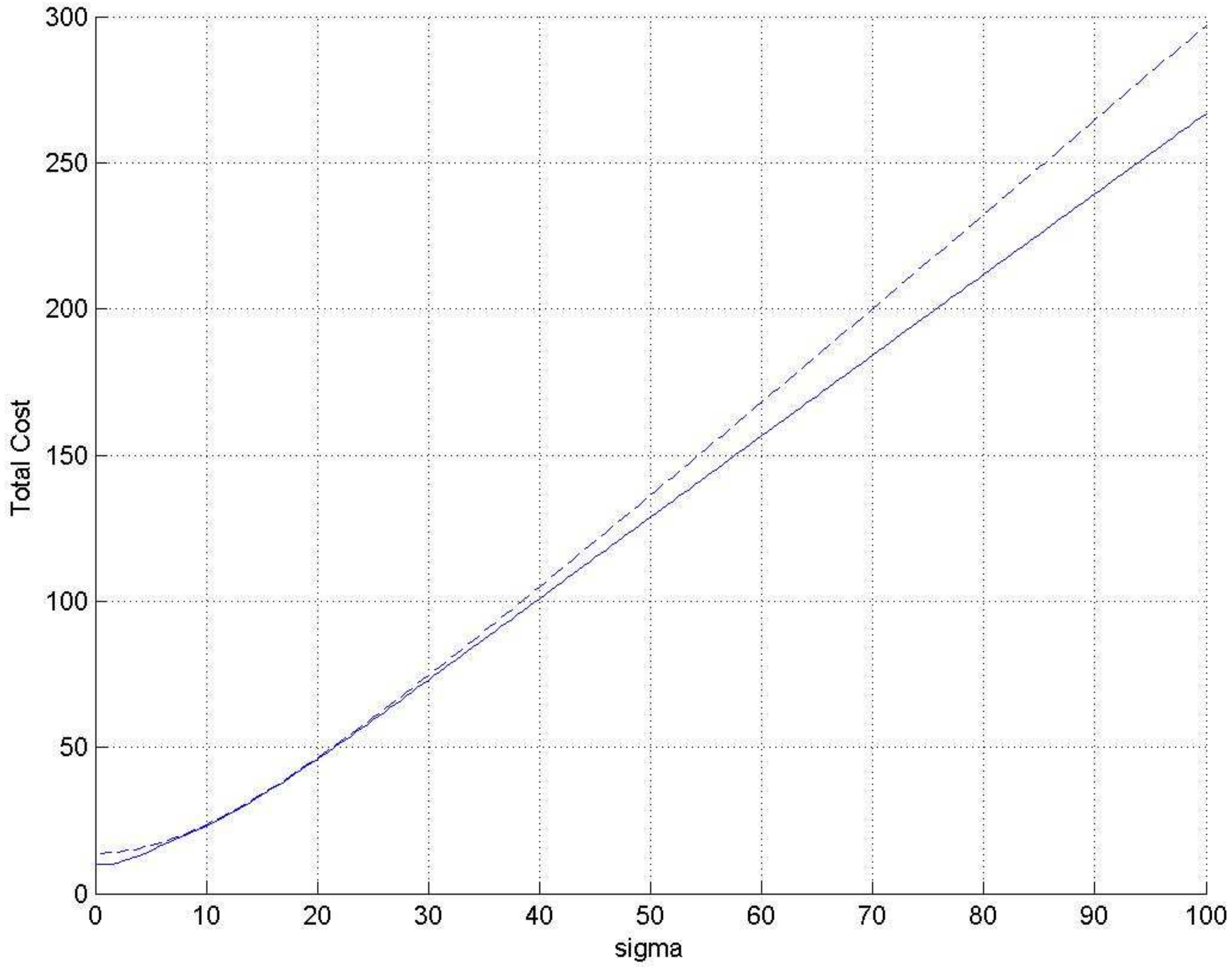

Figure 1. Total Cost
Dashed line is standard model; solid line is drift control model

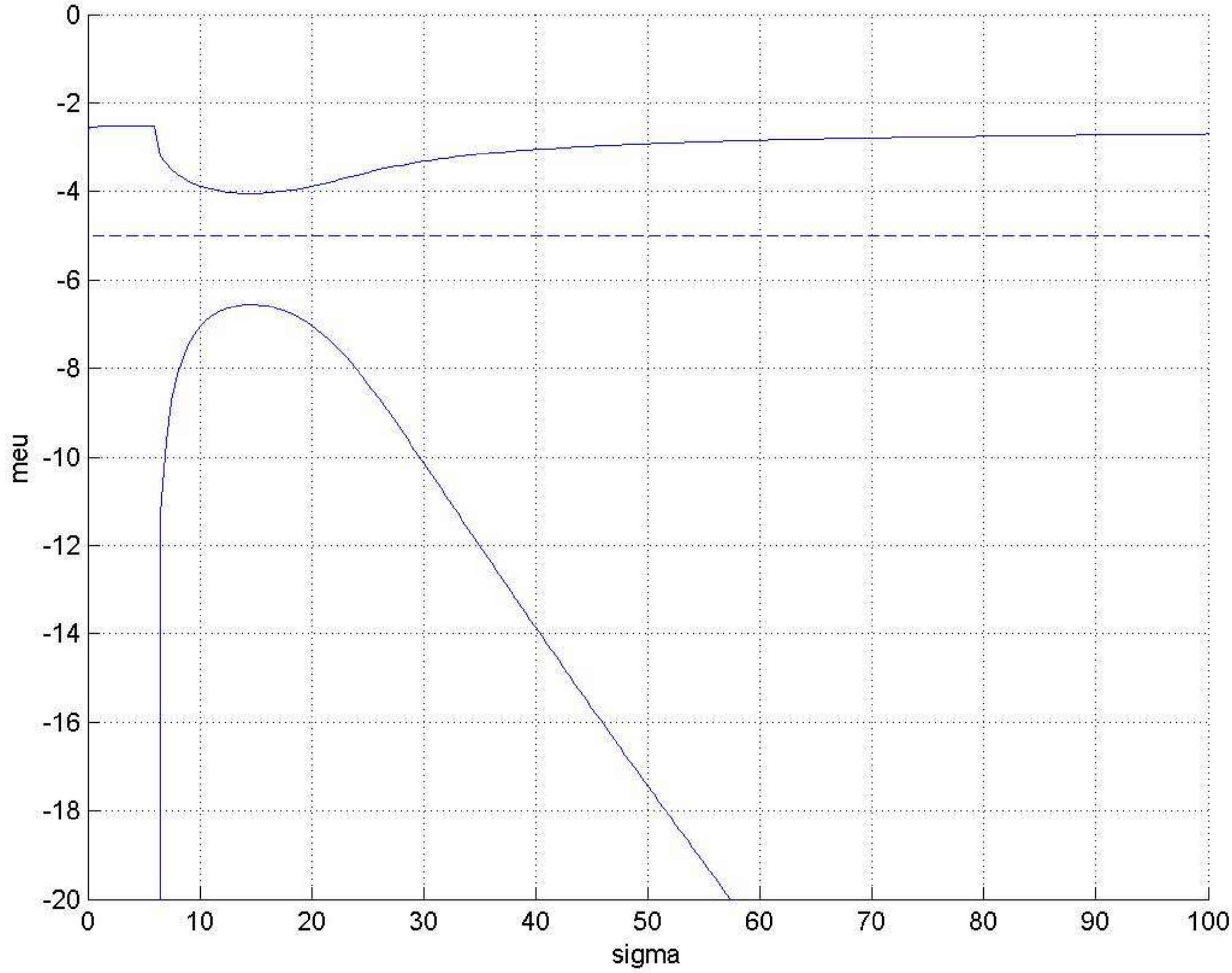

Figure 2. Intra-period consumption rates
Dashed line is standard model; solid lines are drift control model. Bottom solid line is initial consumption rate (μ_0); top solid line is the second rate (μ_1)

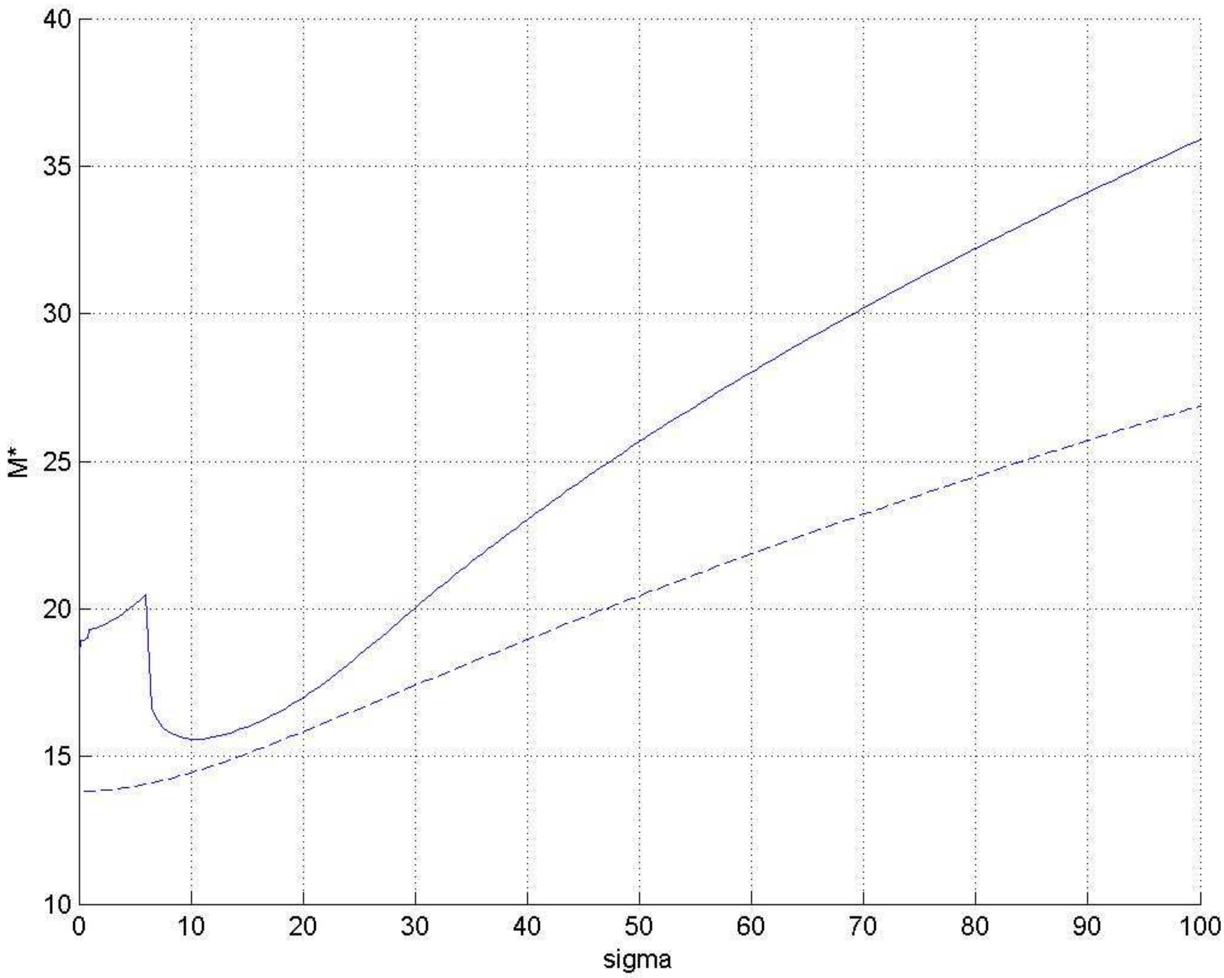

Figure 3. Optimal Cash Withdrawal
Dashed line is standard model; solid line is drift control model

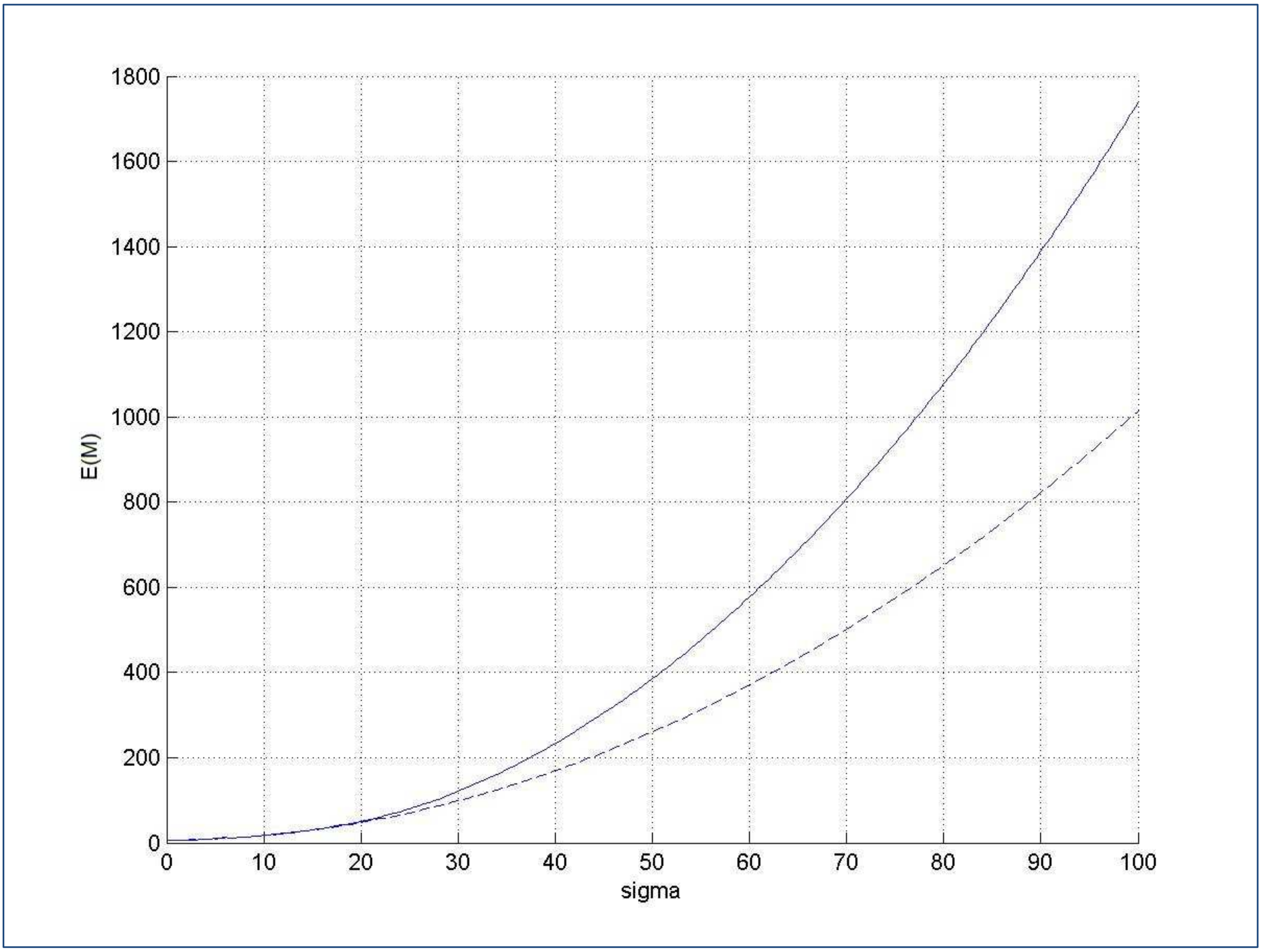

Figure 4. Average Intra-Period Cash Holdings
Dashed line is standard model; solid line is drift control model