Money and the Size of Transactions

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Abstract

Consumers make transactions of different sizes over time. This paper shows that this fact, together with transaction costs of various assets, can help in developing a theory of liquidity. Assets with different cost structures are used to purchase different sizes of transactions. This can explain the demand for money itself, the precautionary demand for money, and the demand for cash and demand deposits. Thus consumers use cash for small transactions, demand deposits for larger transactions, and use savings for the largest transactions. Finally, the paper shows that modeling banks as suppliers of liquidity leads to a better understanding of their success as financial intermediaries.

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1. Introduction

This paper claims that the theory of liquidity can benefit significantly from incorporating the observation that transactions have different sizes. People purchase vegetables in the market on one day, purchase a CD on another day, and buy a car in the following month. Each purchase involves a different size of payment. This simple fact of life together with the fact that assets differ in their transaction costs structure, means that each asset is used to perform a different class of size of transactions. Hence, there are positive demands for assets with different liquidities. This can help in understanding many monetary issues, like why we need money, the precautionary demand for money, the portfolio choice between bonds and money, the emergence of commercial banks, and why banks specialize in financial intermediation.

The paper uses a very simple framework of analysis to demonstrate how the different sizes of transaction shed new light on monetary issues. It assumes that due to taste shocks people consume different amounts in each period of time, which also differ from the amounts they produce and sell, namely their income. That gives rise to a demand for money, as money is used to transfer income over time when other stores of value have very high transaction costs. Money gives us the flexibility to purchase different amounts in different periods of time and this flexibility is what we usually call 'liquidity.' Note that the type of money issued is not discussed, and is only assumed to be a durable and fully divisible asset.

Taste shocks are not known in advance. Hence, people leave aside some money in order to be able to perform large purchases in case of need. This is the
precautionary demand for money. It is a result of future taste shocks and of risk aversion. Hence, this framework can formalize the precautionary demand for money and examine how it is related to various parameters, like the distribution of taste shocks, inflation, etc.

The role of money in transferring income over time becomes less crucial if there are other stores of value like land, capital and bonds, which have higher rates of return. It is shown that money is still held with these assets if these additional assets have transaction costs, as assumed in the well-known Baumol-Tobin analysis. Embedding the transaction cost assumption in the framework of different sizes of transactions leads to the result that small purchases are paid by money, while large purchases are paid from savings. This explains why consumers hold positive amounts of both assets. This Baumol-Tobin result therefore depends crucially on having different sizes of transactions. Otherwise only one of the two assets would be superior. Hence, this paper sheds new light on the Baumol-Tobin approach.

Next, the paper discusses the coexistence of cash and demand deposits and introduces commercial banks to the analysis. Banks introduce an additional transaction technology, demand deposits. Note that holding cash is costly, due to possibility of theft or loss. Deposits eliminate this cost, but create a cost per transaction, either the cost of going to the bank, or the cost of processing a check in a clearing house. As a result people face two competing transaction technologies, with different cost structures. Since future transactions come in all sizes, people hold both cash and deposits, and use cash for small transactions, and demand deposits for larger ones. Hence, this approach can also shed light on the emergence of banks as offering a new transaction technology.

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1 The idea that demand deposits emerge when holding cash is risky appears also in a recent paper by He, Huang and Wright (2004).
The paper then turns to discuss lending by banks of some of their deposits. This raises several issues, like optimal size of reserves, risk of run on the bank, etc. The paper briefly shows that its framework can handle these issues analytically and even contribute to the understanding of why commercial banks are leading financial intermediaries. This is because they obtain funds at a lower cost than other financial intermediaries, as they do not pay interest on deposits. Hence, banks are successful financial intermediaries not because of superior information, but rather because they have cheaper funds, as they pay their depositors with liquidity.

The paper discusses these issues in a series of models with taste shocks and with overlapping generations. The models are quite similar, and differ only because each highlights a different issue. The use of overlapping generations models is for tractability only. A model with infinitely lived consumers, where time periods are shorter, would have been more realistic, but is harder to solve analytically. Appendix A outlines the basic model within a framework of infinitely lived agents.

This paper belongs to the literature that tries to explain the demand for money within a framework of utility maximization. Research in this area has followed a few lines. The first one, beginning with Sidrausky (1967), assumes that money appears in the utility function, as it facilitates transactions and enables consumers to increase leisure. This approach is very popular, due to its tractability, but is vulnerable to the Lucas critique, as utility is implicit and hence changes with policy. The second line of research, which began with Lucas (1980), assumes that money has to be held ahead of time to pay for transactions. It is called "cash-in-advance". Another effort has been made by Romer (1986) and later on by Starr (2003) and others to explicitly embed the Baumol (1952) and
Tobin (1956) models within a Ramsey model. Another approach to the problem has been followed by Wallace (1980) and Bewley (1980, 1983), who focus on money as a store of value that transfers income over time in face of income and taste shocks. The most recent approach to modeling money is the search theory of money that formalizes the tradition of lack of 'double coincidence of wants.' This theory models people who produce and consume different types of goods, who thus have to search for trading partners, and money reduces search costs. These models began with Diamond (1984) and Kiyotaki and Wright (1989, 1991 and 1993).  

The specific contribution of this paper to this literature can be described as combining the approach of Wallace and Bewley with that of Baumol and Tobin by adding the assumption that transactions come in many sizes. Since assets differ in transaction costs each is used for a different set of transaction sizes. The paper is also related to the search theory of money as it sits well in the tradition of money as a way to overcome lack of 'double coincidence of wants.' But here, the mismatch is not of the types of goods transacted, but of their sizes, or values.

The idea of distinguishing between transactions by size in relation to the use of cash and demand deposit has appeared in some papers, Prescott (1978) and Whitesell (1992). But these papers deal with different sizes of goods within the same period and do not extend this idea to difference in size of transaction over time and thus do not relate it to the foundation of the demand for money. In a recent paper Berentsen and Rocheteau (2003) present a search theoretic model of money, where traders differ with how much they favor the good, which is somewhat related to size of transaction.

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2 Recent examples of this literature are Kocherlakota (1998), Wallace (2001), Lagos and Wright (2004), and Rocheteau and Wright (2003).
The structure of the paper is as follows. Section 2 presents the first model, which describes the role of money as a store of value. Section 3 formalizes the precautionary demand for money. Section 4 adds bonds with transaction costs and shows that the demand for money is still positive. Section 5 introduces bank deposits as a new transaction technology. Section 6 discusses bank loans and financial intermediation, while Section 7 summarizes. Appendix A describes a Ramsey version of the model and Appendix B presents proofs to propositions.

2. Model I: The Demand for Money

This is the basic model in the paper, which shows how the role of money as a medium of exchange and its role as a store of value are closely related when transactions differ in size over time. This model shows how money can facilitate transactions and increase welfare so that the demand for it is positive. We show it by using the simplest possible model, where people wish to consume different quantities than what they produce and thus need a medium of exchange, which also stores value over time.

Consider an economy with one physical good, which is used for consumption only. The consumption good is not durable. The economy consists of overlapping generations, each with a continuum of people of size 1. Each person lives 2 periods, works in both and produces a fixed amount \( y \) in each period. The individual consumes in the 2 periods, but consumption can differ over time, due to inter-temporal preferences. The utility function of an individual is:

\[
U = \theta \log c_1 + (1 - \theta) \log c_2,
\]
where $c_1$ is consumption in first period of life, $c_2$ is consumption in second period and $\theta$ is a taste shock, which realizes in first period of life. It is assumed that taste shocks differ across individuals and are uniformly distributed on $[0, 1]$ in each generation.

If the consumption good is not durable each individual consumes exactly $y$ in each period of life. That is true even if barter trade is taking place, if people for some reason cannot consume their own products and trade with others. We next introduce an asset called money, which is durable. Money can be of many forms and types. It can be metal like silver or gold, or fiat money, but the following analysis is the same. Let the outstanding amount of money in period $t$ be denoted by $M_t$ and let the price of the physical good at time $t$ in terms of money be $P_t$. We assume that money is issued by the government, which increases it at a fixed rate $\mu, \mu \geq 0$, namely:

\begin{equation}
M_t = M_{t-1}(1 + \mu).
\end{equation}

The amount of new money issued each period is used by the government to purchase the physical good, at a quantity $M_{t-1}\mu / P_t$.

Equilibrium analysis begins with utility maximization of an individual born in period $t$ with a taste shock $\theta$. The individual sells output $y$, buys first period consumption and keeps by the end of first period of life an amount of money $m_t(\theta)$. This amount is determined by utility maximization and is equal to

\begin{equation}
m_t(\theta) = \arg \max_{m \geq 0} \left\{ \theta \log \left( y - \frac{m}{P_t} \right) + (1 - \theta) \log \left( y + \frac{m}{P_{t+1}} \right) \right\}.
\end{equation}

Note that $P_{t+1}$ is the expected price in the next period and due to rational expectations it is known in period $t$, since aggregates are deterministic in this economy. The constraint $m \geq 0$ is the liquidity constraint of the consumer in the first period of life.
We distinguish between two cases in utility maximization (I.3). In the first case the individual has a low taste shock, thus low consumption in first period of life, and does not reach the liquidity constraint. In the second case the consumer has a high $\theta$, wants to purchase a large amount and the liquidity constraint is binding. The first case is described by point A in Figure 1. In this case, the amount of real balances of money, which is also denoted $l_t(\theta)$, is derived from the first order condition:

$$\frac{m_t(\theta)}{P_t} = l_t(\theta) = y \left(1 - \theta - \theta \frac{P_{t+1}}{P_t}\right).$$

Equation (I.4) shows that the individual demand for real balances depends negatively on the expected rate of inflation. Case A holds as long as individuals choose to keep money for second period, namely as long as:

$$\theta \leq \theta_t \equiv \left(1 + \frac{P_{t+1}}{P_t}\right)^{-1}.$$

Equation (I.5) shows that the set A decreases as the rate of inflation rises. The reason is that it increases the incentive to consume in the present.

[Insert Figure 1 here]

The second case, where $\theta > \theta_t$, is called B and is shown by point B in Figure 1. In this case the liquidity constraint is binding, no money is left and the individual consumes exactly $y$ in each period of life. Clearly, the introduction of money increases the utility of those who are in case A. Hence, this model can justify the introduction of money to the economy. Note that in this model money raises welfare only if it is durable, namely if it stores value.
We next turn to derive the monetary equilibrium. We sum up the demands for money by the young. These are the amounts they keep by the end of the period as the amount of money used for transactions within the period only changes hands and is not counted as net demand for money. Hence, the aggregate demand for real balances is

\[
\frac{M_r \ d}{P_t} = \int_0^1 l_i(\theta) d\theta = \int_0^\theta \left( 1 - \theta - \theta\frac{P_{t+1}}{P_t} \right) d\theta = \frac{y}{2 \left( 1 + \frac{P_{t+1}}{P_t} \right)}. \tag{I.6}
\]

Hence, the demand for real balances depends negatively on the expected rate of inflation, \(P_{t+1} / P_t\). The demand for liquidity is shown in Figure 2.

[Insert Figure 2 here]

To complete the derivation of the monetary equilibrium we add the monetary dynamics to Figure 2, through the horizontal curve at \(1 + \mu\). It is easy to show that the intersection of the two curves is the rational expectations stable equilibrium. The rate of inflation is therefore equal to the rate of monetary expansion and the equilibrium price level in period \(t\) is described by:

\[
P_t = \frac{2(2 + \mu)}{y} M_t. \tag{I.7}
\]

It is interesting to analyze a variant of this benchmark model, where the taste shocks have variable volatility. Consider the same model except that taste shocks are uniformly distributed on \([\frac{1}{2} - \sigma, \frac{1}{2} + \sigma]\) in each generation, where \(0 \leq \sigma \leq \frac{1}{2}\). It can be shown that in this case the demand for real balances is equal to:

\[
\frac{M_r \ d}{P_t} = \frac{y}{4\sigma} \left[ \frac{1}{2} + \sigma - \left( \frac{1}{2} - \sigma \right) \frac{P_{t+1}}{P_t} \right] \left[ \left( 1 + \frac{P_{t+1}}{P_t} \right)^{-1} - \frac{1}{2} + \sigma \right]. \tag{I.8}
\]
The demand for real balances therefore depends negatively on the expected rate of inflation as in the benchmark model, but it also depends negatively on the variability of shocks $\sigma$. Note that in this case inflation is bounded, as demand for money becomes zero above the following rate of inflation: $(\frac{1}{2}+\sigma)/(\frac{1}{2}-\sigma)$.

This section therefore shows how money can be useful when people have different sizes of transactions over time. They use money to transfer income from one period to another if their consumption in first period is low enough. For money to function this way it must be a store of value. Hence, the roles of money as a means of transaction and as a store of value are closely related when transactions differ in size over time. Note that the demand for money depends negatively on the variability of transactions. Broadly speaking, this model of money belongs to the tradition of 'mismatch of wants.' But instead of mismatch of types of goods, here we have a mismatch of values of goods transacted, as the amount sold (income) is different from the amount bought.

3. Model II: The Precautionary Demand for Money

According to our basic approach people use money in transactions when the amount they wish to purchase is larger than the amount they earn as income. Hence, money enables them to use the opportunity to make a larger purchase. The more money they have available, the greater the chance that they will be able to purchase without being liquidity constrained. This leads to a precautionary demand for money. One of the reasons people hold money is to be able to make large unanticipated purchases. Our framework of analysis enables us to deal with this concept in a formal way. Although it is usually
impossible to separate between the precautionary demand and other motives for holding money, we present in this section a simple extension of our model that is capable of that.

Consider a closed economy with only one physical good, which is used for consumption and is perishable. Population consists of overlapping generations, where each generation is a continuum of size 1. Each person lives 3 periods, produces in each period an amount $y$ and consumes in the three periods according to the following utility function:

\begin{equation}
U = \frac{1}{2} \log c_1 + \theta \log c_2 + (1 - \theta) \log c_3.
\end{equation}

The taste parameter $\theta$ differs across people and is independent and uniformly distributed on $[0, 1]$. Furthermore, the taste parameter is stochastic, namely unknown to the individual in first period of life and is revealed only in the second period. We assume that the only asset in the economy is money. Its quantity is $M_t$ and the nominal price of the physical good is $P_t$. In a similar way to model 1 the rate of monetary expansion is fixed and equal to $\mu$.

A person born in $t$ earns $y$ in first period of life, consumes $c_{1,t}$ and keeps her savings in money, with real value $l_{1,t} = y - c_{1,t}$. In second period of life she consumes her income plus her money savings from period $t$, minus $l_{2,t+1}$, the real amount of money left in the end of the second period. In third period of life she consumes income plus money saved from previous period. In the second period of life the individual realizes her taste shock $\theta$ and she maximizes utility for the second and third periods:

\begin{equation}
l_{2,t+1}(\theta) = \arg \max_{l \geq 0} \left\{ \theta \log \left( y + l \frac{P_t}{P_{t+1}} - l \right) + (1 - \theta) \log \left( y + l \frac{P_{t+1}}{P_{t+2}} \right) \right\}.
\end{equation}
This maximization can lead to two possible outcomes. One, denoted A, occurs when the first order condition holds and the consumer has sufficient liquidity. The other, denoted B, holds when the consumer wants to purchase more in the second period and is liquidity constrained. The two cases can be described in a diagram similar to Figure 1. In order to simplify notation we denote from here on the gross rate of inflation by:

$$\Pi_{t+1} = \frac{P_{t+1}}{P_t}.$$

Then, in case A the demand for money is determined by the FOC and is equal to:

$$(II.3) \quad l_{2,t+1}(\theta) = y(1 - \theta - \theta\Pi_{t+2}) + (1 - \theta) \frac{l_{1,t}}{\Pi_{t+1}}.$$

Calculating consumption in second and third periods of life yields the lifetime utility in set A:

$$(II.4) \quad U_{A,t}(\theta) = \frac{1}{2} \log(y - l_{1,t}) + N(\theta) - (1 - \theta) \log \Pi_{t+2} + \log \left[y(1 + \Pi_{t+2}) + \frac{l_{1,t}}{\Pi_{t+1}}\right],$$

where the function $N$ is used for: $N(\theta) = \theta \log \theta + (1 - \theta) \log(1 - \theta)$. The set A includes the sufficiently small taste shocks, for which the demand for money in second period is positive. Hence, the borderline between sets A and B is:

$$(II.5) \quad \theta_{t+1} = \frac{y\Pi_{t+1} + l_{1,t}}{y\Pi_{t+1} + l_{1,t} + y\Pi_{t+1}\Pi_{t+2}}.$$

In set B, where $\theta > \theta_{t+1}$, the consumer faces a liquidity constraint, leaves no money for third period and consumption in second and third periods are $y + \frac{l_{1,t}}{\Pi_{t+1}}$ and $y$ respectively. In this set lifetime utility is
\begin{equation}
U_y(\theta) = \frac{1}{2} \log(y - I_{t,y}) + \theta \log \left(y + \frac{I_{t,y}}{\Pi_{t+1}}\right) + (1 - \theta) \log y.
\end{equation}

Since in first period of life the individual does not know her future taste shock $\theta$, the demand for money in this period is determined by maximizing expected utility. The expected utility is

\begin{equation}
(EU)_t = \int_0^{\theta_{t+1}} U_{A,t}(\theta) d\theta + \int_{\theta_{t+1}}^1 U_{B,t}(\theta) d\theta.
\end{equation}

The following proposition describes the results of the maximization of expected utility in the first period of life.

**Proposition 1:** There is a unique optimal amount of $l_{1,t}$ that maximizes the expected utility (II.7). This amount of money is proportional to income: $l_{1,t} = yh(\Pi_{t+1}, \Pi_{t+2})$, where $h$ depends negatively on the expected rates of inflation in $t+1$ and $t+2$. Furthermore, if inflation is sufficiently low, $l_{1,t}$ is positive, and only for a high inflation it becomes zero.

Proof: See Appendix B.

Hence, individuals in this economy are willing to sacrifice consumption in first period of life in order to reduce the risk of being liquidity constrained in second period of life. We view $l_{1,t}$ as the precautionary demand for money. If the rate of inflation is high, the losses from holding money outweigh the benefits from reducing the risk of liquidity constraint and the consumer reduces the precautionary demand for money to zero.

In order to demonstrate that the demand for money by the young, $l_{1,t}$, is a precautionary demand for money, assume that people have access to a better asset than money: it not only stores value, but also enables them to borrow. In other words, assume
that they have a bond with the same interest rate as money, 0, but that can be used both to save for next period and to borrow in order to purchase in the present. In this case liquidity of grownups is never constrained. It can be shown that in this case the expected utility is maximized at $l_{1,t} = 0$. Hence, the amount of money held by the young in our model with money is purely a precautionary demand for money.

We next turn to derive the equilibrium in the economy. The demand for real balances is the sum of the demand by grownups, which is due to variable sizes of transactions, and the demand by the young, which is the precautionary demand for money. Their sum is:

\[
\frac{M_t^d}{P_t} = l_{1,t} + \int_0^\theta l_{2,t}(\theta)d\theta = \\
= \int_0^\theta h(\Pi_{t+1}, \Pi_{t+2}) + \int_0^\theta [1 - \theta - \theta \Pi_{t+1} + (1 - \theta) \frac{h(\Pi_{t+1}, \Pi_{t+2})}{\Pi_t}]d\theta.
\]

The following proposition describes the equilibrium and how real balances are related to the rate of inflation.

**Proposition 2:** The equilibrium rate of inflation depends on the previous period real balances. At the steady state the rate of inflation is equal to the rate of monetary expansion $\mu$. The demand for money falls with inflation.

**Proof:** See Appendix B.

4. Model III: Money and Bonds

So far money has been the only asset in the economy and hence, it is used to transfer income over time. This section shows that it is also used when there are additional assets
as well, if there are transaction costs. In this section we add another asset to the economy, which can be described either as bonds or as physical capital. For the sake of simplicity assume that this asset has a constant positive real rate of return \( r \). This asset has a higher return than money, but it is less liquid, since it is costly to invest in this asset and to withdraw the investment for use. More specifically, assume that finding an investor and signing a contract has utility cost of size \( x \), and withdrawing the loan after a period or two has a utility cost as well, though smaller, \( z < x \). This assumption follows the Baumol-Tobin tradition, of a costly exchange between bonds and money. Below we discuss the difference between this model and previous models in this tradition.

The rest of the model is similar to the above models. Each person lives three periods, produces an amount \( y \) in the first period of life only, and consumes in the second and third periods of life.\(^3\) The utility from consumption is

\[
U = \theta \log c_2 + (1 - \theta) \log c_3.
\]

The taste shock \( \theta \) is revealed only in the second period of life. We assume for simplicity that the taste shocks are uniformly distributed on \([0, 1]\). As in the previous models we assume that the government increases the amount of money at a constant rate \( \mu \). To simplify notation this section analyzes the steady state of the economy only. The analysis of dynamics is similar and follows Model II. In the steady state the rate of inflation is constant and equal to the rate of monetary expansion, so:

\[
\Pi = 1 + \mu.
\]

We first analyze the representative consumer, who decides not only on consumption levels but also on the composition of her portfolio between money and
bonds. An individual saves in first period of life all income and divides it between money and lending:

\[
y = \frac{m}{P} + b = l + b,
\]

where \( l \) is real balances of money and \( b \) is real amount of bonds.

In the second period of life there can be three cases, which depend on the taste shock \( \theta \). In case A, the transaction size in second period is smaller than money, so that some money is left after the transaction. It is not invested in bonds as we assume that \( x \) is large enough, so it is kept as money for the third period of life. In case B, the size of transaction is larger, so that all the money is used. In case C, the transaction is even larger, so it is paid by all the money and by adding some savings. This is done by going to the investor and retrieving some of the loan back.

In case A the consumer leaves \( l_2 \) real balances of money and keeps it to third period of life. This amount is determined by the following utility maximization:

\[
\max_{l_2} \left\{ \theta \log \left( \frac{l_1}{\Pi} - l_2 \right) + (1 - \theta) \log \left[ b_1 (1 + r)^2 + \frac{l_2}{\Pi} \right] - x - z \right\}.
\]

The optimal amount of money left is therefore:

\[
\begin{align*}
l_2(\theta) &= \Pi^{-1} \left[ (1 - \theta) l_1 - \theta (1 + r)^2 \Pi^2 b_1 \right] = \\
&= \Pi^{-1} \left[ (1 - \theta) l_1 - \theta (1 + r)^2 \Pi^2 + \theta (1 + r)^2 \Pi^2 l_1 \right]
\end{align*}
\]

Calculating the optimal levels of consumption in second and third periods of life we get that case A utility is equal to

\[
U_A(\theta) = N(\theta) - (2 - \theta) \log \Pi + \log \left[ l_1 - l_1 (1 + r)^2 \Pi^2 + y (1 + r)^2 \Pi^2 \right] - x - z.
\]

This assumption is made for simplicity only. Since the main focus of this model is not saving and consumption, but portfolio decisions, the main results of the model are not affected by this assumption.
Case A occurs as long as \( l_2 \) is non-negative, namely if the taste shock is smaller than the threshold \( \theta_1 \):

\[
\theta_1 = \frac{l_i}{y(1 + r)^2 \Pi^2 - [(1 + r)^3 \Pi^2 - 1]l_i}.
\]

In case B the consumer is mildly liquidity constrained. She uses all her money, but is not sufficiently constrained to call off a loan. In this case consumption levels are \( c_2 = l_i / \Pi \) and \( c_3 = b_i (1 + r)^2 \) in the second and third periods respectively and utility is

\[
U(B) = 2(1 - \theta) \log(1 + r) - \theta \log \Pi + \theta \log l_i + (1 - \theta) \log(y - l_i) - x - z.
\]

In case C the taste shock \( \theta \) is larger and so is consumption. The liquidity constraint is so binding that the consumer goes to the investors and retrieves part of the loan, of size \( d \). The consumer maximizes

\[
\theta \log \left( \frac{l_i}{\Pi} + d \right) + (1 - \theta) \log \{b_i (1 + r) - d \}(1 + r) \} - x - 2z.
\]

The amount of loan called off is therefore equal to:

\[
d(\theta) = \theta y (1 + r) - l_i \left[ \theta (1 + r) + \frac{1 - \theta}{\Pi} \right].
\]

Hence, utility in this case is

\[
U(C) = N(\theta) + (1 - \theta) \log(1 + r) - \log \Pi + \log \left\{ y (1 + r) \Pi - l_i ((1 + r) \Pi + l_i) \right\} - x - 2z.
\]

Next note, that the borderline between case B and case C is determined by indifference between the two states, namely by the condition: \( U(B) = U(C) \). This condition boils down to

\[
N(\theta_2) - (1 - \theta_2) \log(1 + r) \Pi + \log \left\{ y (1 + r) \Pi - l_i ((1 + r) \Pi - 1) \right\} - \theta_2 \log l_i - (1 - \theta_2) \log(y - l_i) = z.
\]
This threshold preference level $\theta_2$ is binding if it is smaller than 1. If $U_B$ is everywhere higher than $U_C$ then $\theta_2$ is equal to 1. It is easy to see that this does not happen if $l_1$ is low.

We can next calculate the expected utility of the individual in period 1 of life, which is when portfolio decisions are made:

$EU = \int_0^{\theta_1} U_A(\theta) d\theta + \int_{\theta_1}^{\theta_2} U_B(\theta) d\theta + \int_{\theta_2}^1 U_C(\theta) d\theta.$

The individual determines the amount of money to hold by equating the marginal utility of money to zero or by equating the cost and the benefits of holding money. Increasing $l_1$ is costly as it reduces the future income from bonds. But it is also beneficial, as it reduces the probability of selling bonds and thus reduces the probability of suffering the utility cost of $-z$. In other words, increasing the amount of money reduces the probability of hitting the liquidity constraint. Proposition 3 shows that the individual always prefers to hold a positive amount of money, despite its lower rate of return.

**Proposition 2**: The amount of money held by the young is positive at any rate of inflation. It is also proportional to income: $l_1 = yh$, it is negatively related to the rate of inflation, and it is positively related to $z$. If the transaction cost $x$ is sufficiently high, the consumer holds money only and no bonds, so that: $l_1 = y$.

**Proof**: In Appendix B.

In this specific model the overall demand for money is by the young and by the grownup of case A, who do not use all their money in second period of life. Hence the equilibrium in the money market is determined by the following equilibrium condition:
This condition determines the price level in this economy. It can be shown that the right hand side, which is the demand for real balances, depends negatively on the nominal interest rate \((1 + r)\Pi\).

In this paper money is used to transfer income over time, in order to be able to make transactions of different sizes over time. In model III there is another asset – bonds – that can also do the same job, transfer income over time. This asset has a higher rate of return, but it also has transaction costs. In other words, the two assets have different cost structures. If all transactions were of the same size, then they would all be performed either by money or by bonds, and the two assets would not co-exist. Here the assumption that transactions come in different sizes becomes crucial, as some transactions are better financed by money and some by savings. Thus, there are positive demands for the two assets, money and bonds. Hence, this model follows the Baumol-Tobin tradition, but embeds it in a framework of different sizes of transactions, which clarifies why the Baumol-Tobin assumption works. It is interesting to note that in the original Baumol-Tobin model all purchases are the same, but they are not equal to income, as consumption is done every period while income arrives only once in a while. While this is an implicit assumption in the original models, in this paper it becomes explicit and its role is fully emphasized. This model also clarifies how money and savings are used for different types of purchases, where money pays for smaller transactions, while for larger transactions people pay out of their saving accounts.
5. Model IV: Commercial Banks – Cash and Demand Deposits

This section introduces commercial banks to our framework. Since our approach to money focuses on transactions, we introduce banks as a technological innovation in the area of transactions. Banks offer an alternative means of payment to cash. Instead of holding cash people can put money in the bank in demand deposits. This has a clear benefit of reducing the risk involved with carrying large amounts of money, due to loss or theft. Hence, banks create an additional transaction technology, demand deposits. This is a safer asset, but transactions are costly. The way demand deposits are used by consumers, whether by going to the bank to draw cash for each purchase or by using checks, is less important for our analysis. What is important is that in any way there is a cost involved in the transaction itself. Cash transactions are not costly, but holding cash is costly due to risk of loss or theft. There are therefore two transaction technologies, or two assets, cash and demand deposits, which have different cost structures, similar to money and bonds. In a similar way it is shown that the two forms of money – cash and demand deposits – are held, one is used for small size transactions and the other for large ones. Hence, banks emerge as a way to reduce the cost of holding cash, by offering a different transaction technology, but since this technology is not completely costless and has a different cost structure, banks do not completely crowd out the use of cash, and the two forms of money coexist.

To demonstrate this idea, consider a similar economy to those in previous models. The economy has overlapping generations, each of size 1, and each person lives three periods, produces \( y \) in the first period of life, and consumes in the second and third periods of life where utility is
(IV.1) \[ U = \theta \log c_2 + (1-\theta) \log c_3. \]

The taste shock \( \theta \) is random, it is revealed only in the second period of life, and \( \theta \) is uniformly distributed on \([0, \tau]\), where \( \tau < 1 \). This is assumed for tractability of the solution only. For further simplicity assume that there are no bonds but money only. People can hold cash, which faces risk of theft or loss, or they can deposit money in banks, which are safe. Going to the bank to deposit or withdraw money has utility cost of size \( v \).\(^4\) In order to model the risk of holding cash as simple as possible we assume that if the amount of cash held is too large it is stolen. Namely, if the amount of cash held by a person exceeds a real value \( E \), it is stolen. Since it is reasonable that this threshold of theft is relative to income, we assume that it is equal to \( e y \), where \( e > 0 \). Hence, \( e \) measures danger of theft in the economy.\(^5\) We further assume that \( e < 1 - \tau \). As for monetary policy, we assume as in former models that the government increases the amount of money at a fixed rate \( \mu \). Furthermore, we analyze the steady state only, where the rate of inflation is fixed and equal to the rate of monetary expansion. Hence, the gross rate of interest is: \( \Pi = 1 + \mu \).

A person earns income \( y \) in first period of life and keeps it in two forms of money. A real amount \( f \) is held in cash and a real amount \( d \) is deposited in the bank, so that \( f + d = y \). Demand deposit pays no interest. In the second period of life the person realizes her \( \theta \) and four cases are possible. In case A, a small amount is purchased with some of the cash, and some cash, \( f/\Pi - c_2 \), is left for third period. In case B the transaction is larger and all the cash is used in period 2, but no bank deposit is used to

\(^4\) When people pay with checks this is usually an income loss rather than utility cost, as checks cost money. The cost is different, but the two costs structures are similar and the results of the model are similar as well.
avoid the cost of going to the bank. In the third case C the consumer wishes to consume more in period 2 and both cash and some of the demand deposit are used, but some of the demand deposit is left in the bank. This is done because the anticipated purchase in period 3 is too large to be paid by cash only. In case D all the money in the bank is withdrawn, some is used for purchase and some remains as cash. Since \(1 - \theta\) is larger than \(e\) the consumer is constrained by the threat of theft and keeps only \(ey\) as cash. Hence, small transactions are paid by cash and large transactions by demand deposits. The following proposition describes some characteristics of the equilibrium.

**Proposition 4:** The amount of cash held by each young consumer is \(ey\). Both demands for cash and for demand deposits are proportional to income \(y\). The demand for cash depends positively on the average size of taste shocks \(\tau\), but demand deposits and the overall amount of money depend negatively on it.

**Proof:** In Appendix B.

Analyzing the effect of inflation on the demands for cash and for deposits is more cumbersome. To simplify the calculation we assume that the cost of going to the bank \(v\) is sufficiently high so that case C disappears, as the consumer prefers to keep money in cash for third period rather than go once more to the bank. Hence, \(\theta_2\) is determined by the intersection of \(U_B\) and \(U_D\):

\[
\theta_2 \log e + (1 - \theta_2) \log(1 - e) = \theta_2 \log(1 - e\Pi) + (1 - \theta_2) \log(e\Pi).
\]

(IV.2)

\footnote{An alternative modeling strategy is to assume that cash depreciates at some rate. In this case the model is fully equivalent to model III of money and bonds.}
Note that if there is no inflation, $\Pi = 1$, then $\theta_2 = \frac{1}{2}$. If the rate of inflation rises, $\theta_2$ decreases. Based on the proof to proposition 4 the demand for cash in this case is:

\[ F = ye\left(2 + \frac{e}{2\Pi} - \frac{\theta_2}{\tau}\right). \]

Hence, the effect of inflation on the demand for cash is ambiguous. The demand for bank deposits in the case is:

\[ D = y(1-e)\left(1 + \frac{\theta_2}{\tau}\right). \]

Hence, the demand for bank deposits depends negatively on the rate of inflation. Clearly inflation affects demand deposits by more than it affects cash holdings.

The overall demand for money in this case is:

\[ L = y\left[1 + d + \frac{e^2}{2\Pi} + \frac{\theta_2}{\tau} \frac{1-e-e\Pi}{\Pi}\right]. \]

Hence the demand for money is falling with the rate of inflation.

Note that in this section the outstanding amount of money must be equal to the total amount of cash and demand deposits since banks do not lend yet and as a result they hold a reserve ratio of 100%. In the next section we add lending to the model and examine its effects on money and on financial intermediation.

6. Model V: Banks and Financial Intermediation

The situation described in model IV, where banks have large stocks of cash in their safes, tempts banks to lend some of the money in order to earn interest on it. This leads us to issues of bank lending, optimal reserves, and how these are related to the monetary equilibrium. This section presents a preliminary analysis of these issues, using a model.
similar to the previous one. The main additions to model V are that we bring back bonds or physical capital to the model, to enable bank lending, and we also introduce cost to banking activity. Let us assume that there is a class of investors, who have projects that require investment of 1 in one period. The return on a project in next period, $1 + R$, depends on the overall amount of capital invested:

(V.1) \[ R = R(K), \]

where $K$ is the overall amount of capital invested and $R$ is a decreasing function.\(^6\) This is the return on a project run by a good investor. A bad project yields 0. It is possible to perfectly monitor and screen investors before lending them, but this monitoring requires work of $q$ workers per project selected. This gives rise to financial intermediaries as delegated monitors as in Diamond (1984), and the resulting net rate of return for consumers who lend to these financial intermediaries is: \[ r = R - yq. \]

Consumers come in overlapping generations, each of size 1. Each person lives three periods, earns labor income $y$ in the first period of life, and consumes in the second and third periods of life where utility is

(V.2) \[ U = \theta \log c_2 + (1 - \theta) \log c_3. \]

The taste shock $\theta$ is random, it is revealed only in the second period of life, and $\theta$ is uniformly distributed on $[0, \tau]$, where $\tau < 1$. There is a utility cost of going to the financial intermediary and create a loan contract or withdraw resources from the contract. This utility cost is $x$.\(^7\) In addition to loans or bonds people can hold cash, which faces risk of theft or loss, or they can deposit money in banks, which are safe. Going to the bank to

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\(^6\) It is implicitly assumed that there is an additional fixed factor of production, like land.

\(^7\) We can also assume as in model III that the cost of writing a contract is not the same as withdrawing the money. The results are similar.
deposit or withdraw money has utility cost of size $v$. We assume realistically that $v < x$. Similar to model IV assume that if the amount of cash held is larger than $e y$, it is stolen.

Banks require a fixed investment and labor to run the transactions and the deposits. In order to form a bank an amount $s$ is invested and $n$ workers take care of the following amount of demand deposits:

(V.3) \[ d = an^a. \]

As for monetary policy, assume for the sake of simplicity that the supply of cash is fixed at $M$, so that only a steady state equilibrium with fixed prices is analyzed.

We can now sketch the equilibrium of this economy. In first period of life consumers build a portfolio of three assets: cash, deposits, and savings, in the amounts $f_1, d_1$, and $b_1$ respectively. In second period of life they realize $\theta$, consume, and keep cash, deposits and bonds in the amounts $f_2(\theta), d_2(\theta)$ and $b_2(\theta)$ respectively. Summing up over all individuals we get the overall deposits in the banks in each period: $D = D_1 + D_2$, and the overall withdrawals from banks, which are equal. Hence, if banks are fully diversified in the population, so that the distribution of $\theta$ among their clients is the same as in the population, then they can lend all their deposits and keep zero reserves. What is clear from this description is that banks can lend funds to investors and monitor them, just like other financial intermediaries, but the funds they use are less expensive, as they don't have to pay the interest rate of lenders. They pay zero interest to depositors and only supply them with liquidity. Hence, banks have an advantage in competition with other financial intermediaries.

To illustrate it we examine banks' decisions. A bank lends its deposits $d$ and additional funds it intermediates (saving accounts) $b$. The bank's net profit is:
(V.4) \[(1 + r)(d + b) - yn - (1 + r)(b + s) = (1 + r)an^\alpha - yn - (1 + r)s.\]

Profit maximization leads to the following amount of deposits in the bank:

(V.5) \[d = a^{\frac{1}{1-\alpha}} \left( \frac{\alpha(1 + r)}{y} \right)^{\frac{\alpha}{1-\alpha}}.\]

The optimal profit of the bank is:

(V.6) \[(1 + r)[(1 - \alpha)d - s] = (1 + r) \left( (1 - \alpha)a^{\frac{1}{1-\alpha}} \left( \frac{1 + r}{y} \right)^{\frac{\alpha}{1-\alpha}} \frac{\alpha}{\alpha^{1-\alpha}} - s \right).\]

Note, that as long as the bank's profit is positive, it can offer better deals on saving accounts. As a result it has an advantage over other financial intermediaries. Hence, banks can rule the market for financial intermediation to a large extent. Lending continues until \(K\) is sufficiently large and \(R\) falls so that profits (V.6) become 0. Note, that the edge of banks over other financial intermediaries is not due to any comparative advantage in information processing, but only because banks pay less for funds, for which they pay with the transaction services to depositors.

From the zero profit condition of banks we can derive the equilibrium amount of capital in the economy:

(V.7) \[R(K) = \left( \frac{s}{1-\alpha} \right)^{\frac{1-\alpha}{\alpha}} a^{\frac{1}{\alpha}} \frac{1}{\alpha} \frac{y}{\alpha} + e - 1.\]

The overall amount of bank's deposits and lending is \(D\), as long as it is smaller than \(K\), and that determines the amount of banks in the economy:

(V.8) \[\frac{D}{d} = \frac{D}{s} (1 - \alpha).\]
Next consider an extension of the model that describes lending in a more realistic way, so that banks hold positive reserves. For that we need to add some uncertainty to money holding, or some uncertainty to the distribution of risk. This can be easily modeled within our framework by assuming that the distribution of taste shocks has a random aggregate component. Formally, this is done by assuming that \( \tau_t \) is random and changes from one period to the other. Hence banks need to keep some reserves in order to avoid the risk of illiquidity. We do not fully model it here, but the intuition at least seems to be quite straightforward.

Focusing on liquidity shocks and reserves can highlight another issue in monetary economics, which is how central banks affect real activity. Note that central banks lend to banks who are hit by liquidity shocks, in this model \( \tau_t \). The interest rate of this lending is lower than the market interest rate. Hence, this can be viewed as a subsidy to financial intermediation by banks. By changing the interest rate of lending to banks the central bank affects the cost of funds to banks and thus affects the interest rate they charge for their loans. Thus such a policy has real effects and not only nominal effects. The intuitive reason for that is that such a change in the interest rate by the central bank changes not only the stock of money, but the flow of income from the bank to the economy, or to the commercial banks. In a way, this approach views monetary policy as part of fiscal policy, where the central bank directly subsidizes the banking sector and through it investors in the economy. Note, that this view on the effect of monetary policy does not require any assumption of price rigidity. Monetary policy affects the economy as it directs real resources to some sectors in the economy.
7. Summary

This paper suggests that the best way to understand the issue of liquidity is to combine the idea of transaction costs with the idea that transactions come in different and random sizes. Assets that differ in their transaction costs can be held in the portfolio only if transactions have different sizes. Otherwise only one type of asset is dominant. Thus the combination of the two ideas can serve for a foundation to the theory of liquidity and therefore to the theory of money. Thus people perform small transactions with cash, larger transactions with credit cards and demand deposits and for larger transactions they withdraw money from their saving accounts.

This paper shows how this idea can be used to understand a number of issues in monetary economics, like precautionary demand for money, cash and deposits, financial intermediation by banks, etc. But this is just a beginning of the development of this idea. One way to proceed is to continue the analysis of banks and their liquidity. Another direction of research is to fully introduce a central bank and discuss monetary policy beyond the discussion in Section 6. Another way to proceed is to apply the more realistic Ramsey model from Appendix A and solve it at least numerically. Calibration methods can help to estimate transaction costs of various assets.
Appendix A: The Infinite Horizon Model

Consider an economy with a single physical good. Time is discrete. There is a continuous mass of size 1 of infinite horizon individuals in the economy. Each one produces a constant amount $y$ in each period of time. Individuals derive utility from consumption, but they have taste shocks in each period, that affect their utility. The utility from consumption in period $t$ of individual $j$ is:

\begin{equation}
E_t \sum_{s=0}^{\infty} \theta^s u[c_{t+s}(j), \theta_{t+s}(j)].
\end{equation}

The utility function is concave in consumption and satisfies the Inada conditions. The taste shocks $\theta$ are independent and identically distributed over time and across individuals. Each shock is revealed in its period, but future taste shocks are unknown in advance. Assume that output is produced by labor only and there is no capital.

If there is no money either, consumers cannot shift income over time to meet their different taste shocks and must consume $y$ in each period. The utility of consumer $j$ in period $t$ is therefore equal to:

\begin{equation}
E_t \sum_{s=0}^{\infty} \theta^s u[y, \theta_{t+s}(j)].
\end{equation}

Next assume that there is money in the economy and people can use it for trade. Assume that money is durable, coins or fiat money. Hence, it can be used to transfer income over time. Assume that the overall amount of money in the economy in period $t$ is $M_t$. Let us denote the amount of money held by individual $j$ by the end of period $t$ by $m_t(j)$. Then the budget constraint in period $t$ is described by:
The consumer maximizes utility (A.1) given the budget constraints (A.3) and the non-negativity of money constraint:

(A.4) \[ m_t(j) \geq 0. \]

Clearly, the optimal utility with money is higher than utility without money (A.2). Hence, the money enables consumers to adjust better to taste shocks, namely to make transactions of different sizes over time.

We next turn to describe equilibrium in this economy. The equilibrium prices are the sequence \((P_t, P_{t+1}, \ldots)\) of nominal prices of the physical good in terms of money. As shown below these prices are fully known in advance, namely prices are deterministic. The consumers react to these prices as they determine the rates of return of money. Hence, the individual maximizes (A.1), given the budget constraints (A.3) and (A.4) and taking the price levels as given.

It can be shown that the optimal real amount of money held by end of period \(t\) is described by a function of initial money, taste shock, and expected rates of inflation:

(A.5) \[
\frac{m_t(j)}{P_t} = l\left(\frac{m_{t-1}(j)}{P_t}, \theta_t(j), \pi_t\right),
\]

where \(\pi_t\) is the vector of all future rates of inflation:

(A.6) \[
\pi_t = \left(\frac{P_t}{P_{t+1}}, \frac{P_{t+1}}{P_{t+2}}, \ldots\right).
\]

Equation (A.5) describes the individual demand for money. Note that this demand for money can be positive, if the constraint (A.4) is not binding, namely if the consumer does not consume much in period \(t\) and can leave money to future consumption. But if a taste
shock raises the marginal utility of consumption in t the constraint (A.4) becomes binding and the consumer does not leave money for future periods.

Equation (A.5) also describes how the distribution of money across individuals evolves over time. Note that since the taste shock is independent across individuals it is also independent of the amount of money from period t-1. Hence, the distribution of money in period t is determined uniquely by the distribution in t-1, namely it evolves deterministically over time.

The aggregate demand and the aggregate supply of money determine the equilibrium in the market for money, which also determines the price of goods \( P_t \):

\[
\frac{M_t}{P_t} = \int_{0}^{1} \left( \frac{m_{t-1}(j)}{P_t}, \theta(j), \pi_t \right) dj.
\]

Since the taste shock is independent of the initial distribution of money from period t-1, it follows that the equilibrium price level is fully determined by the distribution of money in t-1 and is therefore not stochastic. Hence, present and future prices are fully determined by the initial distribution of money, and by future amounts of money.

In order to complete the proof of existence and uniqueness of equilibrium (which is only sketched here) note that the price levels in all periods depend on the path of future prices as shown by equations (A.5) and (A.7). A fixed-point argument shows the existence and uniqueness of an equilibrium price path. Note, that the equilibrium price is finite in each period. Namely money has a positive value despite the fact that it does not produce any direct utility. If for example there are no taste shocks the price of money is equal to zero and it has no value. To see this note that in this case all those who hold money in period t want to consume it since they have preference to the present (as \( \beta < 1 \)). As a result the demand for goods exceeds \( y \), but since the equilibrium amount of
consumption must be \( y \) in this exchange economy, the equilibrium real balances become 0, namely money has no value. Hence, money has value in this economy only because transactions have different sizes.

The model can be extended by adding an asset that has higher return than money but is less liquid, as in the Baumol-Tobin approach and as done in Section 4 in the paper. If bonds pay a fixed real rate of interest \( r \), but each sale of bonds has a fixed cost of real size \( x \), people hold money none the less, for making small purchases. If the purchase is big enough, people might be willing to bear the cost \( x \) and not lose utility as a result of not making the full purchase. In a similar way all the main ideas of the paper can be developed within this framework as well, but the solution is less tractable, and possibly only numerical and not analytical. Hence, the paper uses a much simpler version, where life cycles are short and the model is solved fully.

Appendix B: Proofs

Proof of Proposition 1:

Derivation of (II.7) with respect to \( l_{1,t} \) yields

\[
\frac{\partial EU_t}{\partial l_{1,t}} = \frac{\partial \theta_1}{\partial l_{1,t}} \left[ U_A(\theta_{t+1}) - U_B(\theta_{t+1}) \right] + \int_0^{\theta_{t+1}} \frac{\partial U_A(\theta)}{\partial l_{1,t}} d\theta + \int_{\theta_{t+1}}^1 \frac{\partial U_B(\theta)}{\partial l_{1,t}} d\theta.
\]

Since the first item in the RHS is zero, as utility at the threshold between A and B is equal, the marginal utility of first period money is

\[
\frac{\partial EU}{\partial l_1} = -\frac{1}{2} y - l_1 + \theta_{t+1} \left[ \frac{1}{y \Pi_{t+1} + y \Pi_{t+2} + l_{1,t}} + \frac{1}{2} \frac{1 - \theta_{t+1}^2}{y \Pi_{t+1} + l_{1,t}} \right]
\]

\[
= \frac{1}{2} y \left[ -1 + \frac{1}{\Pi_{t+1} + h} + \frac{\Pi_{t+1} + h}{(\Pi_{t+1} + \Pi_{t+2} + h)^2} \right].
\]

(B.1)
where:

(B.2) \[ h = \frac{I_t}{y}. \]

(B.1) is decreasing with liquidity \( h \) and as \( h \) approaches 1 it goes to \(-\infty\). Hence a maximum exists and is unique. It is determined by the \( h \) that equates (B.1) to 0, or by \( h = 0 \) if (B.1) is negative already at zero. Furthermore, liquidity in first period of life is proportional to income as \( l_{1,t} = yh \). It can be shown that (B.1) is negatively related to both \( \Pi_{t+1} \) and \( \Pi_{t+2} \). Hence, an increase in the rate of inflation reduces \( h \) and reduces the precautionary demand for money. Next examine when \( h \) is positive. When liquidity \( h \) is equal to zero, the marginal expected utility is equal to:

\[
\frac{1}{2y} \left[ -1 + \frac{1 + (1 + \Pi_{t+2})^2}{\Pi_{t+1}(1 + \Pi_{t+2})^2} \right].
\]

Hence, the precautionary demand for money is positive, \( h > 0 \), if:

(B.3) \[ (\Pi_{t+1} - 1)(1 + \Pi_{t+2})^2 < 1. \]

Namely, if the rate of inflation is not too high this condition holds and the young hold a positive amount of money for precautionary reasons. If inflation is high (B.3) does not hold and the precautionary demand is zero. Q.E.D.

Proof of Proposition 2:

Calculation of the demand for money (II.8) and using (II.5) we get that the equilibrium in the money market is described by:

(B.4) \[ \frac{M_t}{P_t} = y \left[ h(\Pi_{t+1}, \Pi_{t+2}) + \frac{(\Pi_t + h(\Pi_t, \Pi_{t+1}))^2}{2\Pi_t(\Pi_t + \Pi_t \Pi_{t+1} + h(\Pi_t, \Pi_{t+1}))} \right]. \]
The dynamics of real balances are described by:

\[
\frac{M_t}{P_t} = \frac{M_{t-1}}{P_{t-1}} 1 + \mu \Pi_t.
\]

From equating (B.4) and (B.5) we get that the equilibrium rate of inflation satisfies:

\[
\Pi_t = F\left(\frac{M_{t-1}}{P_{t-1}}, \Pi_{t-1}, \Pi_{t+1}\right).
\]

Hence, the rational expectations equilibrium rate of inflation is a function of previous real balances:

\[
\Pi_t = \Pi\left(\frac{M_{t-1}}{P_{t-1}}\right).
\]

It can be shown that with this equilibrium function the steady state is stable. In the steady state the rate of inflation is equal to the rate of monetary expansion: \( \Pi_t = 1 + \mu \). Real balances at the steady state are equal to:

\[
\frac{M}{P} = y \left[ h + \frac{(1 + \mu + h)^2}{2(1 + \mu)(2 + 3\mu + \mu^2 + h)} \right].
\]

It can be shown that real balances depend negatively on \( \mu \), directly and through \( h \). Q.E.D.

Proof of Proposition 3:

The marginal expected utility with respect to first period money satisfies:

\[
\frac{\partial EU}{\partial l_1} = \frac{\partial \theta_1}{\partial l_1} \left[U_A(\theta_1) - U_B(\theta_1)\right] + \frac{\partial \theta_2}{\partial l_1} \left[U_B(\theta_2) - U_C(\theta_2)\right] + \\
+ \int_0^\theta \frac{\partial U_A(\theta)}{\partial l_1} d\theta + \int_0^\theta \frac{\partial U_B(\theta)}{\partial l_1} d\theta + \int_\theta^1 \frac{\partial U_C(\theta)}{\partial l_1} d\theta =
\]

\[
= \int_0^\theta \frac{\partial U_A(\theta)}{\partial l_1} d\theta + \int_0^\theta \frac{\partial U_B(\theta)}{\partial l_1} d\theta + \int_\theta^1 \frac{\partial U_C(\theta)}{\partial l_1} d\theta.
\]
Hence, the marginal expected utility is equal to:

\[
\frac{\partial EU}{\partial l_1} = \frac{\theta_2^2 - \theta_1^2}{2} \frac{1}{l_1} - \theta_1 \frac{(1 + r)^2 \Pi^2 - 1}{y(1 + r)^2 \Pi^2 - l_1[(1 + r)^2 \Pi^2 - 1]} - \frac{(1 - \theta_1)^2 - (1 - \theta_2)^2}{2} \frac{1}{y - l_1} - \frac{(1 - \theta_2)}{(1 + r) \Pi - 1} \frac{(1 + r) \Pi - 1}{y(1 + r) \Pi - l_1[(1 + r) \Pi - 1]}. \tag{B.6}
\]

Marginal expected utility is decreasing in \( l_1 \). We next show that the marginal expected utility at \( l_1 = 0 \) is infinite, and hence the optimal amount of money, where the marginal expected utility is zero, must be positive. To show this note that as \( l_1 \to 0 \) both \( \theta_1 \) and \( \theta_2 \) approach zero as well. We next show that the first item in the RHS of (B.6) goes to \( \infty \) when \( l_1 \to 0 \), while it is easy to see that the other terms are negative and finite. From the definition of \( \theta_2 \) in (III.10) we get that as \( l_1 \) and \( \theta_2 \) converge to zero we get:

\[
\theta_2 \log l_1 \xrightarrow{l_1 \to 0} -z.
\]

Hence, for \( l_1 \) close enough to 0:

\[
\theta_2 \geq -\frac{z}{2 \log l_1}.
\]

Hence:

\[
\frac{\theta_2^2}{l_1} \geq \frac{z^2}{4(\log l_1)^2 l_1} \xrightarrow{l_1 \to 0} \infty.
\]

Since from (III.6) we get:

\[
\frac{\theta_1^2}{l_1} \leq \frac{l_1}{(y - l_1)^2 (1 + r)^4 \Pi^4} \xrightarrow{l_1 \to 0} 0.
\]

Namely the first item in the RHS of (B.6) goes to infinity and so does marginal expected utility, as money diminishes to zero. Hence the optimal amount of money must be positive.
From (B.6) and from the definitions of $\theta_1$ and $\theta_2$ we can see that $l_1$ is proportional to income $y$ and hence: $l_1 = yh$. We next show that $h$, namely the demand for money by the young, depends negatively on $(1+r)\Pi$, the nominal interest rate. Using the following notation for the gross nominal interest rate, $I = (1+r)\Pi$, we can write the expected utility as:

$$\text{EU} = \max_{0} \left[ \frac{N(\theta) - (2-\theta) \log \Pi + \log[I^2 - h(I^2 - 1)]}{2(1-\theta) \log(1+r) - \theta \log \Pi + \theta \log h + (1-\theta) \log(1-h)}, \frac{N(\theta) + (1-\theta) \log(1+r) - \log \Pi + \log[I - h(I - 1)]}{-x} \right] + \log y - z - x. \quad \text{(B.7)}$$

Clearly a rise in the nominal interest rate $I$ reduces the marginal utility of holding $l_1$ and hence it reduces the optimal $l_1$.

Since optimal $l_1$ is proportional to $y$, it follows from (B.7) that the optimal expected utility can be written as:

$$\max \text{EU} = \log y + \phi(r, \Pi, z) - x. \quad \text{(B.8)}$$

Note that if the cost of finding an investor and signing a contract is high, the consumer might prefer not to hold bonds at all. In that case utility is equal to:

$$\int_{0}^{1} \left[ N(\theta) - (2-\theta) \log \Pi \right] d\theta + \log y. \quad \text{(B.9)}$$

Hence, if $x > \phi(r, \Pi, z) - \int_{0}^{1} [N(\theta) - (2-\theta) \log \Pi] d\theta$, the consumer prefers to hold money only and does not go to the bonds market. QED.

**Proof of Proposition 4:**

Consider the four cases in second period of life. In case A the consumer purchases consumption with only part of her cash. Utility is:
\[
\theta \log c_2 + (1 - \theta) \log \frac{f + d - c_2}{\Pi} - 2v = \theta \log c_2 + (1 - \theta) \log \frac{y - c_2}{\Pi} - 2v.
\]

Optimization leads to: \( c_2 = \frac{\theta y}{\Pi}, \ c_3 = \frac{(1 - \theta)y}{\Pi^2} \), and cash held for next period is: \( \frac{f - \theta y}{\Pi} \). Utility is:

\[
U_A(\theta) = N(\theta) + \log y - (2 - \theta) \log \Pi - 2v.
\]

In case B consumption is larger, so all the cash is consumed and all demand deposits are kept for next period. Hence:

\[
U_B(\theta) = \theta \log f + (1 - \theta) \log (y - f) - (2 - \theta) \log \Pi - 2v.
\]

In case C consumption is financed by all the cash and by some demand deposits, while some demand deposit are kept in the bank. The consumer maximizes:

\[
\theta \log c_2 + (1 - \theta) \log \frac{y/\Pi - c_2}{\Pi} - 3v.
\]

Hence consumption in second period is \( c_2 = \frac{\theta y}{\Pi} \), and in third period is \( c_3 = \frac{(1 - \theta)y}{\Pi^2} \), and utility is:

\[
U_C(\theta) = N(\theta) + \log y - (2 - \theta) \log \Pi - 3v.
\]

In case D the consumer leaves only cash for third period. Assume that inflation is not too high so that: \( e \Pi < 1 - \tau \). In this case the consumer leaves the maximum amount of cash, namely \( ey \). Hence: \( c_2 = \frac{(1/\Pi - e)y}{\Pi} \) and \( c_3 = ey/\Pi \). Utility in this case is:

\[
U_D(\theta) = \theta \log \left( \frac{1}{\Pi - e} \right) + (1 - \theta) \log e + \log y - (1 - \theta) \log \Pi - 2v.
\]

Note that the borderline between cases A and B is \( \frac{\theta y}{f/y} \). Denote the borderline between B and C (or D if C is empty) by \( \theta_2 \). Note that the utilities for A, C and D do not depend on \( f \). Hence the marginal expected utility is equal to:
\[
\frac{1}{\tau} \int_{\theta_1}^{\theta_2} \frac{\partial U_{\theta}(\theta)}{\partial f} d\theta = \frac{1}{\tau} \int_{\theta_1}^{\theta_2} (\theta - \frac{1 - \theta}{f - y - f}) d\theta.
\]

Note that for \( \theta \geq \theta_1 = f / y \), the integral is positive. Hence, the young consumer increases the amount of cash as much as possible up to the safety bound, so that \( f = ey \).

Given the initial demand for cash we can calculate the amounts of cash in the second period of any individual and then the overall amount of cash. First we calculate the borderlines between the various sets. The borderline between B and C is given by:

\[(B.10) \quad N(\theta_2) - \theta_2 \log e - (1 - \theta_2) \log (1 - e) = v.\]

The borderline between C and D is given by:

\[(B.11) \quad N(\theta_3) - \theta_3 \log (1 - \Pi e) - (1 - \theta_3) \log (\Pi e) = v.\]

The overall amount of cash in the economy \( F \) is:

\[(B.12) \quad F = ye + \frac{y}{\Pi} \int_{\theta_1}^{\theta_2} (e - \theta)d\theta + \frac{y}{\tau} \int_{\theta_1}^{\theta_2} ed\theta = ye \left( \frac{e}{2\tau\Pi} + 2 - \theta_3 \right).\]

Hence, the amount of cash is proportional to income \( y \). Furthermore it depends positively on \( e \), namely on the degree of safety in holding cash. As for \( \tau \), the distribution of taste shocks, it clearly has a negative effect on the demand for cash (note that \( \theta_3 \) does not depend on \( \tau \)).

The overall amount of demand deposits is:

\[(B.13) \quad D = y(1 - e) + \frac{y}{\Pi} \int_{\theta_1}^{\theta_2} (1 - e)d\theta + \frac{1}{\tau} \int_{\theta_1}^{\theta_2} \frac{y(1 - \theta)}{\Pi} d\theta =
= y \left( 1 - e + \frac{y\theta_1}{\Pi} - \frac{ey\theta_2}{\Pi} - \frac{y}{\tau} \frac{\theta_1^2 - \theta_2^2}{2} \right).\]

Note that demand deposits depend negatively on \( \tau \). Q.E.D.
References


Figure 1
Figure 2

\[ P_{t+1}/P_t \]

\[ \pi_{\text{max}} \]

\[ 1+\mu \]

\[ \frac{y}{[2(2+\mu)]} \]

\[ M_t/P_t \]